Memorial University Of Newfoundland

Department of Physics and Physical Oceanography

Physics 3821

EXAMPLE Mid-Term Test

YOU MUST DO ALL FOUR QUESTIONS

Each Question has a Total Value of 10 Points

NOTE: There are no delta function/distribution questions on this example test BUT, you will be expected to know that material and there will be such a question on your midterm!

NAME:

QUESTION 1

i) [2] Write out the Laplace transform. Explain how it is used to solve differential equations. How does it simplify the analysis, and what sort of problems is it well suited to solve.

ii) [] Find the Laplace transform for a sawtooth function amplitude 1 and period T that is;

$$g(t) = \begin{cases} t/T, & 0 \leq t < T; \\ (t-T)/T, & T \leq t < 2T; \\ & \vdots \end{cases}$$

(Major hint: for a periodic function g(t) that repeats with period T, the Laplace transform is

$$F(s) = \frac{G(s)}{1 - e^{-sT}}$$

where $G(s)$ is the transform of $g(t)$ on the interval $0 \leq t < T$.

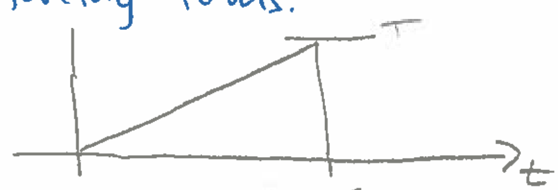
i) $$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\, dt$$ 2

To solve de's with the Laplace transform you take the transform of the equation and then solve an algebraic equation in the transform space. To find the solution you must take the inverse transform of the result. 3

Simplification → turns problem into algebraic problem

→ works well for problems with impulsive forcing terms.

ii) Here $g(t) \longrightarrow$



$$\frac{1}{T}\int_0^T \underset{du}{e^{-st}} \cdot \underset{v}{t}\, dt = -\frac{1}{sT} e^{-st} \cdot t \Big|_0^T - \frac{1}{T}\int_0^T -\frac{1}{s} e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-sT} - \frac{1}{Ts^2} e^{-st} \Big|_0^T$$

$$G(s) = -\frac{1}{s}e^{-sT} - \frac{1}{Ts^2}e^{-sT} + \frac{1}{Ts^2}$$

$$F(s) = \frac{\frac{1}{s^2}\left(-Ts\,e^{-sT} - e^{-sT} + 1\right)}{T\left(1-e^{-sT}\right)}$$

$$F(s) = \frac{1}{Ts^2}\left[\frac{-Ts - 1 + e^{sT}}{e^{sT} - 1}\right]$$

$$\int_0^T \overset{dV}{e^{-st}} \, \overset{U}{\frac{t}{T}} \, dt = \frac{-1}{s} e^{-st} \cdot \frac{t}{T} \Big|_0^T - \int_0^T \frac{-1}{sT} e^{-st} \, dt$$

$$= \frac{-1}{s} e^{-st} - \left[\frac{1}{s^2 T} e^{-st} \right]_0^T$$

$$= \frac{-1}{s} e^{-st} - \left[\frac{1}{s^2 T} e^{-sT} - \frac{1}{s^2 T} \right]$$

$$= \frac{T(-1s)}{T s^2} e^{-st} - \frac{1}{s^2 T} e^{-sT} + \frac{1}{s^2 T}$$

$$= \frac{-(+Ts+1) e^{-sT} + 1}{s^2 T}$$

QUESTION 2

i) [] The Gamma function can be represented by the integral expression:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$$

i) [] Demonstrate that this defnition (correctly) gives $\Gamma(1) = 1$.

ii) [] Now show that this definition meets the requirement that

$$\Gamma(x) = (x-1)\Gamma(x-1)$$

iii) [] And, finally show that for integer values of x this recursion relation reduces to

$$\Gamma(n+1) = n!$$

i) $\int_0^{\infty} e^{-t} dt = -e^{-t}\Big|_0^{\infty} = -(0-1) = \underline{\underline{1}}$ ✓

ii) $\Gamma(x) = \int_0^{\infty} \underset{du}{e^{-t}} \; \underset{v}{t^{x-1}} \, dt$

by parts

$$= -e^{-t} t^{x-1}\Big|_0^{\infty} - \int_0^{\infty} -e^{-t} (x-1) t^{(x-1)-1} dt$$

$$0 - 0 \quad + \int_0^{\infty} e^{-t} (x-1) t^{(x-1)-1} dt$$

$$= (x-1) \cdot \int_0^{\infty} e^{-t} t^{(x-1)-1} dt = \underline{(x-1)\,\Gamma(x-1)}$$

iii) By using answer ii) with $x = n$

$$\Gamma(n) = (n-1) \cdot \Gamma(n-1) = (n-1)\cdot(n-2)\cdot \Gamma(n-3)$$

$$= (n-1)\cdot(n-2)\cdot(n-3)\cdots 1$$

$$= (n-1)! \implies \Gamma(n+1) = n!$$

QUESTION 3

The convolution theorem states that:

$$\mathcal{L}^{-1}(ER) = \mathcal{E} * r = \int_0^t \mathcal{E}(\tau)\, r(t-\tau)\, d\tau$$

i) [3] Describe conceptually what this integral represents. In your answer, comment on what the convolution $\mathcal{E} * r$ equals when $\mathcal{E}(t) = \delta(t)$. What name is given to $r(t)$?

ii) [4] If you solve for the capacitors charge in an rlc circuit you find that:

$$r(t) = \frac{1}{L\omega} e^{-\alpha t} \sin \omega t$$

If this circuit is driven with a sine wave ($\mathcal{E}(t) = A\sin(\Omega t)$), what will $Q(s) = \mathcal{L}\big(q(t)\big)$ be?

iii) [3] Use the convolution integral to find an expression for $q(t)$, the charge on the capacitor in terms of $r(t)$ and $\mathcal{E}(t)$. Don't evaluate the integral, you could use Mathematica to do that!

Useful relations:

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}$$

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}(e^{\alpha t}) = \frac{1}{s - \alpha}$$

$$\mathcal{L}\big(e^{-\alpha t} f(t)\big) = F(s + \alpha)$$

Q 3

i) The convolution integral represents a "sum" over time of the response of a system to all inputs
3 that have come before that time. If the input is a delta function, then you recover the "impulse" response of the system. $\Sigma * r = r(t)$ The output can be considered a weighted sum over the impulse response.

ii) $$r(t) = \frac{1}{L\omega} e^{-\alpha t} \sin \omega t$$

4 $$R = \mathcal{L}[r(t)] = \frac{1}{L\omega} \frac{\omega}{(s+\alpha)^2+\omega^2} = \frac{1}{L}\left[\frac{1}{(s+\alpha)^2+\omega^2}\right]$$

$$E = \mathcal{L}[\Sigma(t)] = \frac{A\Omega}{s^2+\Omega^2}$$

$$\boxed{Q(s) = \frac{A\Omega}{L} \frac{1}{s^2+\Omega^2} \cdot \frac{1}{(s+\alpha)^2+\omega^2}}$$

iii) $$q(t) = \Sigma * r = \int_0^t A \sin(\Omega t) \cdot \frac{1}{L\omega} e^{-\alpha(t-\tau)} \sin\omega(t-\tau)\, d\tau$$

3

QUESTION ~~3~~ 4

The general inverse of the Laplace transform is given by the Mellin inversion integral:

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)e^{st}ds$$

1 i) [] What determines the choice of γ?

7 ii) [] Prove (or should I say demonstrate) that the Mellin integral determines $f(t)$ given $F(s)$.

2 iii) [] Use the Mellin inversion integral to invert

$$F(s) = \frac{s}{s^2 + 2s + 3}$$

(Gift: In this inversion there are certain regions of integration that must go to zero. You must identify these regions but don't need to prove that they go to zero.)

i) γ must be chosen so that all poles of $F(s)$ are to the left of the line $Re(s) = \gamma$

ii) $$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st} \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(\sigma)\, e^{\sigma t}\, d\sigma\, dt$$

If this is $F(s)$ then Mellin works!

exchange $\int$'s

$$= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(\sigma) \int_0^\infty e^{-st} e^{\sigma t}\, dt\, d\sigma$$

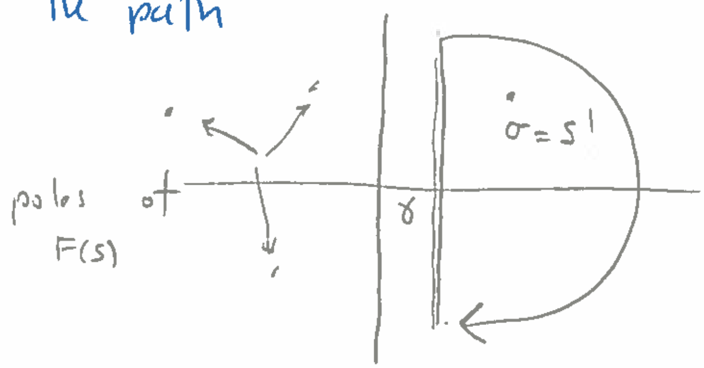
$$= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(\sigma) \left.\frac{e^{(\sigma - s)t}}{\sigma - s}\right|_0^\infty$$

If $Re(s) > Re(\sigma) = \gamma$ $\quad e^{-(s-\sigma)t} \to 0$ as $t \to \infty$

$$= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{F(\sigma)}{(s-\sigma)} \, d\sigma$$

– from integral limits absorbed here!

By construction, $F(\sigma)$ is analytic everywhere to right of ~~...~~ $s = \gamma$ so we can evaluate the complex integral using the path



Now, so long as $F(\sigma) \to 0$ as $R \to \infty$ $\left(R \sim R^{-\varepsilon} \right)$

integral is cw!

$$\mathcal{L}[F(t)] = -\frac{1}{2\pi i} \cdot 2\pi i \underbrace{\lim_{\sigma \to s} (\sigma - s) \frac{F(\sigma)}{(s-\sigma)}}$$

cancel sign

residue at $\sigma = s$

$$= F(s)$$

iii) Invert $F(s) = \dfrac{s}{s^2 + 2s - 3} = \dfrac{s}{(s+3)(s-1)}$

2

Two simple poles $s = -3$ and $s = 1$

$$f(t) = \frac{1}{\cancel{2\pi i}} \cdot \cancel{2\pi i} \cdot \left[\frac{-3}{-4} e^{-3t} + \frac{1}{4} e^{t} \right]$$