Memorial University Of Newfoundland

Department of Physics and Physical Oceanography

Physics 4820

Mid-Term Test

Wednesday, March 13, 2019 | Time: 45 minutes

DO ANY THREE QUESTIONS

(You're grade will be based on the highest 3 marks if you do all 4 questions)

Each Question has a Total Value of 10 Points

NAME:

QUESTION 1

The motion of a string suspended between two fixed points is a Sturm-Liouville problem. It is described by the equation:

$$\nu^2 \frac{\partial^2 Y}{\partial x^2} = \frac{\partial^2 Y}{\partial t^2}$$

where $Y(x,t)$ describes the position of the string as a function of position and time, and $\nu = \sqrt{\tau/\mu}$ (the wave speed), and τ is string tension, and μ is the string linear mass density.

i) [2] Use separation of variables to show that this can be expressed as two coupled differential equations:

$$\frac{d^2 X}{\partial x^2} + k^2 X = 0 \tag{1}$$

and

$$\frac{d^2 T}{\partial t^2} + k^2 \nu^2 T = 0 \tag{2}$$

with $X = X(x)$, and $T = T(t)$.

ii) [1] If the string is clamped at positions $x = 0$, and $x = L$, what are the suitable boundary conditions?

iii) [4] Find the eigenfunctions and eigenvalues for this problem.

iv) [3] What is the equation of motion for the string as a function of x and t if the initial position of the string is given as:

$$Y(x, 0) = 3\,\sin(2\pi x/L)$$

QUESTION 2

i) [3] Show that for a distribution $\phi(x)$,

$$\int_{-\infty}^{\infty} \phi'(x)f(x)dx = -\int_{-\infty}^{\infty} \phi(x)f'(x)dx$$

You can use this result in **question 3** where because the delta function is a distribution,

$$\int_{-\infty}^{\infty} \delta'(x)f(x)dx = -f'(0)$$

ii) [3] If $\phi(x)$ is a distribution, by inspection, which of the following are also valid distributions and which are not. Provide a reason for your choices (for example, in answer to the question is 0×1 finite, a reasonable answer would be to say yes, it equals 0).

a) $\phi(ax)$

b) $\phi(x-a)$

c) $a\phi(x)$

d) $\phi(ax)\delta(y)$

e) $\delta(\sin(x))$

f) $\delta(x)\phi(x)$

iii) [4] Recalling that:

$$\int_{-\infty}^{\infty} \delta(g(x))f(x) = \frac{\sum_{i=1}^{N} f(x_{0i})}{|g'(x_{0i})|}$$

Evaluate the integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-|x|}\delta(x^2+2x-3)dx$$

QUESTION 3

i) [3] Provide a brief definition for the following terms;

Test Function

Core Function

Weak Convergence

Distribution

ii) [3] Prove that a distribution multiplied by an infinitely differentiable function is itself a distribution.

iii) [4] What distribution is represented by the relation $\sin(x)\,\delta'(x)$? Compare that to the distribution represented by $\cos(x)\,\delta'(x)$. (Question 2 tells you what $\delta'(x)$ does!)

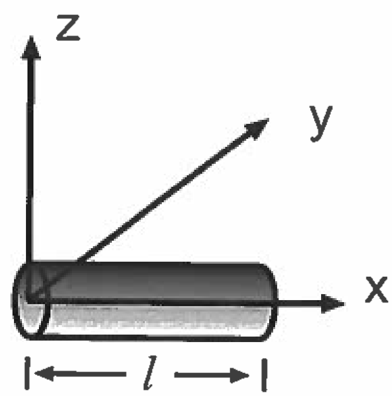
QUESTION 4

A rod of length l, infinitesimal cross section, and mass M lies along the $x-$axis with one end at the origin as shown below. Express the density in terms of delta functions in:

i) [2] Cartesian coordinates.

ii) [4] Cylindrical coordinates.

iii) [4] Spherical coordinates.



(You may want to use step functions as well to constrain the result.)

(And, $dV = dx\,dy\,dz$ or $= \rho\,d\rho\,dz\,d\phi$ or $= r^2\sin\theta\,dr\,d\theta\,d\phi$.)

1

a)
$$v^2 \frac{\partial^2 Y}{\partial x^2} = \frac{\partial^2 Y}{\partial t^2}$$

$$Y = XT$$

$$T \cancel{v^2} \frac{\partial^2}{\partial x^2} X = \frac{1}{v^2} \frac{\partial^2 T}{\partial t^2} X$$

$$\frac{\cancel{v^2}}{X} \frac{\partial^2 X}{\partial x^2} = \frac{\partial^2 T}{\partial t^2} \cdot \frac{1}{T v^2} = -k^2$$

$$\boxed{\cancel{v^2} \frac{\partial^2 X}{\partial x^2} = -k^2 X}$$

2

$$\boxed{\frac{\partial^2 T}{\partial t^2} = -v^2 k^2 T}$$

(must be <0, if $+k^2$, allows exponential solu's but 0 boundary condition is only possible with a trivial $X=0$ solution.)

b) k is the separation constant

c) ① $\boxed{Y(0,t) = Y(L,t) = 0}$

$$\frac{\partial^2 X}{\partial x^2} = -k^2 X$$

$$X = a_n \sin k_n x + b_n \cos k_n x$$

* $b_n = 0$ because $\cos 0 = 1$

$$\boxed{X = a_n \sin k_n x}$$

✓ 2

1 cont.

$$X(L) = 0, \qquad a_n \sin k_n \cancel{L} = 0$$

$$\boxed{k_n L = \frac{n\pi}{\cancel{L}}}$$

$$\boxed{k_n = \frac{n\pi}{L}} \quad \longleftarrow \text{eigenvalues} \quad (2)$$

$$\boxed{X_n = \sin n\pi x/L}$$

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what about T

$$\frac{\partial^2}{\partial t^2} = v^2 k^2 T$$

$$\Rightarrow \quad T = a_n \sin v k_n t + b_n \cos v k_n t$$

$$k_n = \frac{n\pi}{L}$$

→ choose a_n & b_n to meet initial condition ⇒

— $t = 0$

$$Y(x,0) = 3 \sin \frac{2\pi x}{L}$$

I cont

Only eigen function is

$\underline{\underline{n = 2}}$

$$k_2 = \frac{2\pi}{L}$$

$$Y = X_2 T_2 = \left(\cancel{a_n \sin v \cdot \frac{2\pi}{L} t} + \underbrace{b_n \cos v \frac{2\pi}{L} t}_{3}\right) \sin \frac{2\pi x}{L}$$

$$b_n \cos \frac{2\pi}{L} v t \Big|_{t=0} = 3 \Rightarrow \boxed{b_n = 3}$$

$$\boxed{Y = \underbrace{3 \cos \frac{2\pi}{L} v t}_{T} \; \underbrace{\sin \frac{2\pi x}{L}}_{X}}$$

3

Question 2 i)

Show that for a distribution

$$\int_{-\infty}^{\infty} \phi'(x)\, f(x)\, dx = -\int_{-\infty}^{\infty} \phi(x)\, f'(x)\, dx$$

Integrate by parts

$$= \phi(x)\, f(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi(x)\, f'(x)\, dx$$

goes to 0 on strength of test function

$$= -\int_{-\infty}^{\infty} \phi(x)\, f'(x)\, dx \quad !$$

And, since δ is a distribution

$$\int_{-\infty}^{\infty} \delta'(x)\, f(x)\, dx = -\int_{-\infty}^{\infty} \delta(x)\, f'(x)\, dx = -f'(0)$$

Question 2 ii)

a) $\phi(ax)$ Yes $\rightarrow$ general property of distributions

b) $\cancel{\phi}$ $\phi(x-a)$ – Yes really just a variable change

c) $a\,\phi(x)$ – Yes general property of distribution

d) $\phi(ax) * \delta(y)$ – Yes ... variables are different,
~~notice dangerous construction with $\delta(y)$ in denominator~~

e) $\delta(\sin(x))$ – Yes δ of a function

f) $\delta(x)\,\phi(x)$ – No ... products of distributions not generally a distribution

Quiz 2 iii)

Evaluate the integral

$$\int_{-\infty}^{\infty} e^{-|x|} \quad \delta\,(x^2+2x-3)\,dx$$

$$\delta\left[(x-1)\;(x+3)\right]$$

$$\frac{\delta(x-1)}{\left|\,(2x+2)\Big|_{x=1}\right|} + \frac{\delta(x+3)}{\left|\,(2x+2)\Big|_{x=-3}\right|}$$

$$\frac{\delta(x-1)}{|4|} + \frac{\delta(x+3)}{|-4|}$$

$$\rightarrow \int_{-\infty}^{\infty} e^{-|x|} \cdot \left[\frac{\delta(x-1)}{4} + \frac{\delta(x+3)}{4}\right] dx$$

$$= \frac{e^{-1}}{4} + \frac{e^{-3}}{4} = \underline{\underline{0,10442}}$$

3 i)

Test Function: Infinitely differentiable and goes to "0" fast as $x \to \infty$

Core Function: $g(x)$ is infinitely differentiable

Weak Convergence: the sequence of core functions $g_n(x)$ converges weakly if

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} g_n(x) f(x) dx$$ exists for any test function

Distribution: $\phi(x)$ is a distribution if there exists a sequence of core functions that converges weakly to $\phi(x)$

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} g_n(x) f(x) dx = \int \phi(x) f(x) dx$$

3ii)

Let
$$\int_{-\infty}^{\infty} \phi(x)\, f(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)\, f(x)\, dx$$

and $\phi(x)$ is a valid distribution

infinitely differentiable

$$\int_{-\infty}^{\infty} h(x) \cdot \phi(x)\, f(x)\, dx$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} h(x) \cdot g_n(x)\, f(x)\, dx$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)\, h(x) \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \phi(x)\, \underbrace{h(x)\, f(x)}\, dx$$

works as a test function!

$\Rightarrow$ $h(x)\, \phi(x)$ is a valid distribution!

$\longrightarrow$ ✗

3 iii E

What distribution is represented by

$$\sin x\, \delta'(x)$$

$$\int_{-\infty}^{\infty} \sin(x)\, \delta'(x)\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \delta'(x)\, \sin x\, f(x)\, dx = -\frac{d}{dx}\left[\sin(x)\, f(x)\right]\Big|_{x=0}$$

$$= -\cos(x)\, f(x) - \sin(x)\, f'(x)\Big|_{x=0}$$

$$\boxed{= -f(0)} = -\delta(x)$$

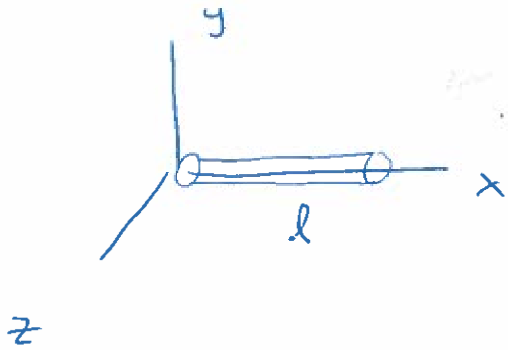
In contrast

$$\int_{-\infty}^{\infty} \delta'(x) \cos x\, f(x)\, dx = -\frac{d}{dx}\left[\cos x\ f(x)\right]_0$$

$$= \sin x\ f(x) - \cos x\ f'(x)\Big|_{x=0}$$

$$\boxed{= -f'(0)} = \delta'(x)$$

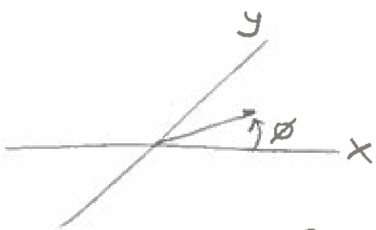
4



a) The rod is restricted to $y=0$ and $z=0$ and exists from $0 - l$ in x

$$\rho(x,y,z) = \frac{M}{l}\,\delta(z)\,\delta(y)\left[\Theta(x) - \Theta(x-l)\right]$$

b) Now in cylindrical coordinates again the rod is restricted to $z=0$ and $\phi=0$



$$y = r\sin\phi$$

$$\delta(y) \to \delta(r\sin\phi) = \frac{1}{r}\,\delta(\sin\phi)$$

$$= \frac{1}{r}\,\frac{1}{|\cos\phi|}\,\delta(\phi) = \frac{\delta(\phi)}{r}$$

giving

$$\rho(r,\phi,z) = \frac{M}{l r}\,\delta(z)\,\delta(\phi)\left[\Theta(r) - \Theta(r-l)\right]$$

4c

~~6~~c)

in spherical coordinates, the rod must have the

$$\rho(r, \theta, \phi) = A\, \delta(\theta - \tfrac{\pi}{2})\, \delta(\phi) \left[\Theta(r) - \Theta(r-\ell)\right]$$

To find A integrate over a spherical shell of radius ρ to $\rho + d\rho$ $\quad \rho < \ell$

In this case $\quad dM = \cancel{\frac{\ell d\rho}{M}}\ \frac{M}{\ell}\, d\rho$

$$\frac{M}{\ell}\, d\rho = \int_0^{\pi}\int_0^{2\pi} A\, \delta(\theta - \tfrac{\pi}{2})\, \delta(\phi) \left[\Theta(r) - \Theta(r-\ell)\right] r^2 \sin\theta\, d\theta\, d\phi\, d\rho$$

$$= A \cdot r^2 \sin\theta \Big|_{\theta = \frac{\pi}{2}} d\rho = A r^2\, d\rho$$

$$A \Rightarrow \quad A = \frac{M}{\ell r^2}$$

$$\boxed{\rho = \frac{M}{\ell r^2}\, \delta(\theta - \tfrac{\pi}{2})\, \delta(\phi) \left[\Theta(r) - \Theta(r-\ell)\right]}$$