Memorial University Of Newfoundland

Department of Physics and Physical Oceanography

Physics 4820

Mid-Term Test

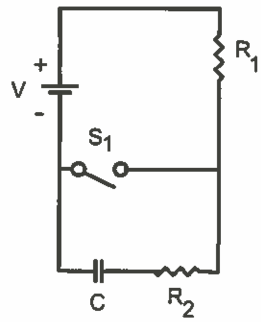
Wednesday, February 9, 2011 Time: 45 minutes

YOU MUST DO ALL FOUR QUESTIONS

Each Question has a Total Value of 10 Points

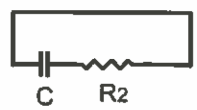
NAME:

QUESTION 1



In the circuit shown, the switch S1 has been closed for a long time and then it is opened.

i) [2] While the switch is closed, the effective circuit for the capacitor becomes:



What is the initial charge on the capacitor?

ii) [1] Given that

$$\frac{\mathrm{d}q}{\mathrm{d}t} = i$$

where i is current and q is charge. How are I and Q (the Laplace transform of i and q) related?

iii) [1] Use Kirchoff's laws to show that the voltage balance for the circuit after the switch has been opened is:

$$V = iR_1 + iR_2 + \frac{q}{C}. \tag{1}$$

iv) [2] What is the Laplace transformed version of (1)?

v) [4] Solve for the CURRENT through the circuit after the switch has been opened.

Useful relations (YOU DON'T NEED THEM ALL):

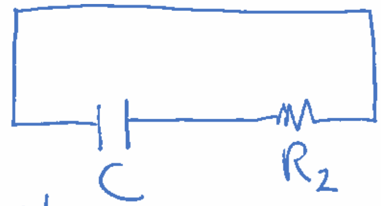
$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}(e^{-\alpha t}f(t)) = F(s+\alpha)$$

$$\mathcal{L}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) = -f(0) + sF(s)$$

1)



i) When the switch is closed, there is a connection between both sides of the ~~resistor~~ ~~two~~ capacitor through a resistor. After a "long time" the charge on the capacitor will be drained.

$$q(t=\infty) = 0$$

2

ii) In general $\mathcal{L}\left[\frac{df}{dt}\right] = -f(0) + sF(s)$

$$\Rightarrow \mathcal{L}\left[\frac{dq}{dt}\right] = -q(0) + sQ = \mathcal{L}[i] = I$$

but since $q(0) = 0$ in this case

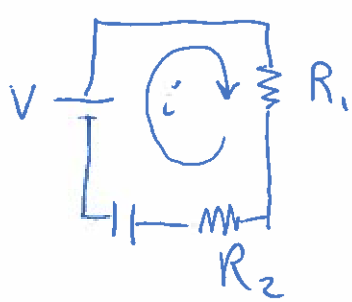
$$\boxed{\begin{array}{c} sQ - q(0) = I \\ sQ = I \end{array}}$$

or

1

iii) After the switch is ~~closed~~ opened the circuit becomes

1



Adding voltage around the loop:

$$V - iR_1 - iR_2 - \frac{q}{C} = 0$$

or

$$\boxed{V = iR_1 + iR_2 + \frac{q}{C}}$$

iv) $\mathcal{L}\left[V = i(R_1+R_2) + \frac{q}{C} \right]$

$$\mathcal{L}[V] = \boxed{\frac{V}{s} = I(R_1+R_2) + \frac{Q}{C}} \leftarrow \text{okay}$$

$\mathcal{L}[i] = I \qquad \mathcal{L}[q] = Q$

or could eliminate Q using answer to ii)

$$\boxed{\frac{V}{s} = I(R_1+R_2) + \frac{I}{sC}}$$

2

v) solve for I in solution to iv ...

$$V = I(R_1+R_2)s + \frac{I}{C} = I\left((R_1+R_2)s + \frac{1}{C}\right)$$

$$\boxed{\frac{V/(R_1+R_2)}{s + 1/C(R_1+R_2)} = I}$$

notice the form $\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-\alpha t} f(t)$

$$\boxed{i(t) = e^{-t/(R_1+R_2)C} \cdot \frac{V}{R_1+R_2}}$$

4

QUESTION 2

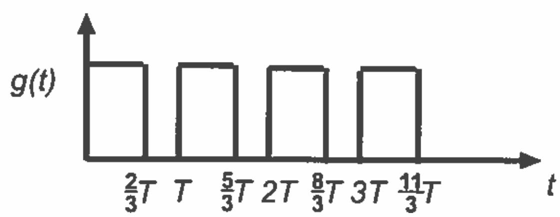
i) [5] Show that the Laplace transform of a function $f(t)$ that is made up of segments $g(t)$ repeated with period T is given by:

$$F(s) = \frac{G(s)}{1 - e^{-sT}}$$

where $\mathcal{L}[g(t)] = G(s)$.

ii) [3] Find the Laplace transform for the modified boxcar function given as:

$$g(t) = \begin{cases} 1, & 0 < t \leq 2T/3; \\ 0, & 2T/3 < t \leq T; \\ \\ 1, & T < t \leq 5T/3; \\ 0, & 5T/3 < t \leq 2T; \\ \\ 1, & 2T < t \leq 8T/3; \\ 0, & 8T/3 < t \leq 3T; \\ \\ & \vdots \end{cases}$$



iii) [2] How is the result changed when the "duration" of the pulse is increased from $2/3\,T$ to T?

②

$$f(t) = \begin{cases} g(t) & 0 - T \\ g(t-T) & T - 2T \\ g(t-2T) & 2T - 3T \\ \vdots & \end{cases}$$

Let

$$\mathcal{L}\left[g(t)\right] = G(s)$$

$$\mathcal{L}\left[g(t-T)\right] = e^{-sT} G(s)$$

$$\mathcal{L}\left[g(t-2T)\right] = e^{-2sT} G(s)$$

$$\vdots$$

$$\mathcal{L}\left[f(t) = g(t) + g(t-T) + g(t-2T) + \cdots\right]$$

$$F(s) = G(s) + e^{-sT} G(s) + e^{-2sT} G(s) + \cdots$$

$$= G(s)\left[1 + e^{-sT} + e^{-2sT} + \cdots\right]$$

$$\boxed{F(s) = G(s) \frac{1}{1 - e^{-sT}}}$$

$$(1 + \alpha + \alpha^2 + \cdots) = \frac{1}{1-\alpha}$$

$\alpha < 1$

ii) $$g(t) = \begin{cases} 1 & 0 \leq t < \frac{2}{3}T \\ 0 & \frac{2}{3}T \leq t < T \end{cases}$$

$$\mathcal{L}\left[g(t)\right] = \int_0^\infty e^{-st}\, g(t)\, dt$$

$$= \int_0^{2/3\,T} 1 \cdot e^{-st}\, dt + \int_{2/3\,T}^{T} 0 \cdot e^{-st}\, dt$$

$$= \int_0^{2/3\,T} e^{-st}\, dt = \frac{-1}{s}\, e^{-st} \Big|_0^{2/3\,T}$$

$$= -\frac{1}{s}\, e^{-\frac{2}{3}sT} + \frac{1}{s}$$

$$= \frac{1}{s}\left[-e^{-2/3\,Ts} + 1\right]$$

And using the repeating function relation

$$\boxed{\mathcal{L}\left[f(t)\right] = \frac{\frac{1}{s}\left(1 - e^{-\frac{2}{3}sT}\right)}{1 - e^{-sT}}}$$

iii)

If the pulse duration is increased to T, $\mathcal{L}[f(t)]$ becomes

$$\mathcal{L}[f(t)] = \frac{\frac{1}{s}\cancel{(1-e^{-sT})}}{\cancel{(1-e^{-sT})}}$$

$$= \frac{1}{s}$$

Laplace transform of a constant value!

QUESTION 3

i) [1] What are the defining characteristics of a delta function? (specifically: what value does it have where and what do you get when you integrate it?)

ii) [2] Use the properties that you identified in part i) to prove the sifting property for the delta function.

iii) [2] What is a delta sequence? Why are they a useful construction?

iv) [2] How is the "block" delta sequence represesented as an equation. Also show a sketch of the progression of this sequence. (If you didn't memorise it just think what it has to do! Well, isn't that better than "just" memorising it anyway?).

v) [3] Use the "block" delta sequence to find what $\delta(ax)$ represents.

i) $\delta(x) = 0 \quad x \neq 0$

goes to ∞ at $x=0$.

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

1

ii) $$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = \int_{-\epsilon}^{\epsilon} \delta(x) f(x)\, dx$$

$\delta(x) \to 0$ away from $x=0$

mean value therem

$$= f(x_0) \int_{-\epsilon}^{\epsilon} \delta(x)\, dx = f(x_0) \int_{-\infty}^{\infty} \delta(x)\, dx$$

$$= f(0) \quad \text{in} \lim_{\epsilon \to 0} \; ! \qquad -\epsilon < x_0 < \epsilon$$

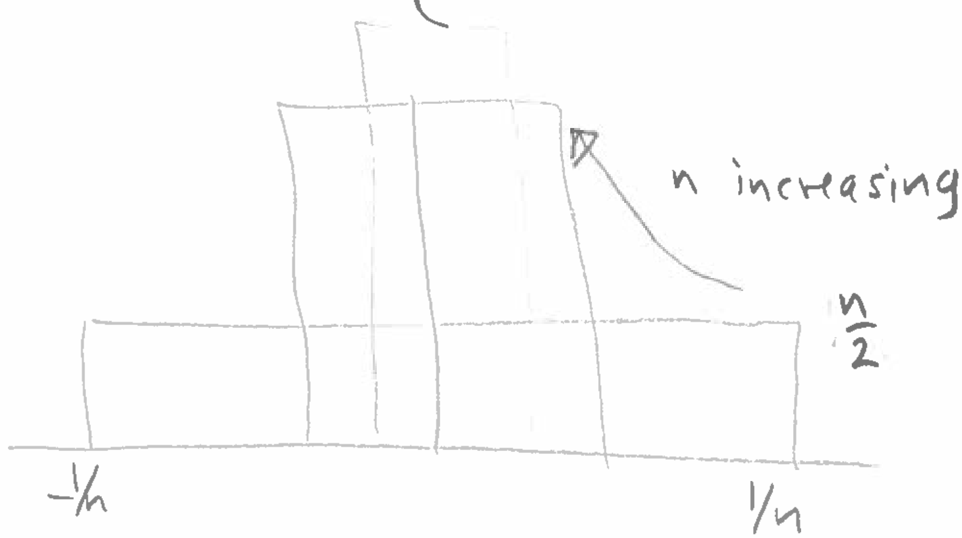
δ function squeezes ϵ infinitesimally small

iii) A delta sequence is a sequence of functions the exhibits the sifting property of the δ function as $n \to \infty$. The δ sequence also has the property that $\phi_n(x) \to 0$ as $x \to \pm\infty$.

The delta sequences are actual functions so that they can be manipulated unlike the δ function.

iv) Step δ sequence

$$\phi_n = \begin{cases} \frac{n}{2} & -\frac{1}{n} < x < \frac{1}{n} \\ 0 & \text{else} \end{cases}$$



v) $$\int_{-\infty}^{\infty} \delta(ax)\, f(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(ax) f(x) dx$$

assume $a > 0$ if $a < 0$

let $u = ax$ $du = a\,dx$ $u = -a'x$ $a' > 0$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(u)\; f\left(\frac{u}{a}\right) \frac{1}{a}\, du$$

cancel

$$\int_{\infty}^{-\infty} \phi_n(u)\, f\left(\frac{-u}{a'}\right) \frac{(-)1}{a}\, du$$

$$= \frac{1}{a} \lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(u)\; f\left(\frac{u}{a}\right) du$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} \delta(u)\; f\left(\frac{u}{a}\right) du$$

$$= \frac{1}{|a|} f(0)$$

QUESTION 4

The integral

$$f(t) = \int_0^\infty \frac{\sin tx}{x}\,\mathrm{d}x \tag{2}$$

is actually a representation of a step function and this can be demonstrated by using a Laplace transform approach!

i) [7] Take the Laplace transform of (2). (Just do it, you will need to change the order of integration and think Laplace).

You should be able to show that:

$$f(t) = \int_0^\infty \frac{\sin tx}{x}\,\mathrm{d}x = \frac{\pi}{2} \qquad t > 0$$

ii) [2] What do you get for $t < 0$ (yes, you could guess but I won't give you any marks unless you show me why)?

iii) [1] Okay, pull it all together and write out the answer for $t < 0$, $t = 0$, and $t > 0$.

Useful relations:

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\mathcal{L}(a) = \frac{a}{s}$$

4]

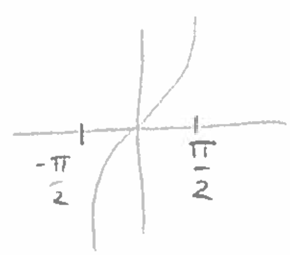
$$F(t) = \int_0^\infty \frac{\sin tx}{x}\,dx$$

$$\mathcal{L}[F(t)] = \int_{\substack{0\\t}}^{\infty} e^{-st} \int_{\substack{0\\x}}^{\infty} \frac{\sin tx}{x}\,dx\,dt$$

Change Integral order

$$= \int_{\substack{0\\x}}^{\infty} \frac{1}{x}\,dx \underbrace{\int_{t} e^{-st} \sin tx\,dt}_{\frac{x}{s^2+x^2}\,!}\,dx$$

$$= \int_0^\infty \frac{1}{\cancel{x}} \cdot \frac{\cancel{x}}{s^2+x^2}\,dx = \frac{1}{s}\tan^{-1}\frac{x}{s}\Bigg|_0^\infty$$



$$= \frac{1}{s}\,\frac{\pi}{2} - \frac{1}{s}\cdot 0$$

$$\boxed{F(s) = \frac{1}{s}\cdot \pi/2} \longrightarrow \mathcal{L}^{-1}[F(s)] = f(t) = \frac{\pi}{2}$$

ii) If $t<0$, let $t' = -t$ $t'>0$

$$\sin tx = \sin -t'x = -\sin t'x$$

$$f(t) = \int_0^\infty \frac{-\sin t'x}{x}\,dx$$

Nothing else changes so that

$$f(t) = \pi/2$$
$$f(-t) = -\pi/2$$

at $f(0)$, $\sin 0 = 0$ so $\underline{f(0) = 0}$

$$f(t) = \begin{cases} \pi/2 & 0 < t < \infty \\ 0 & t = 0 \\ \frac{-\pi}{2} & -\infty < t < 0 \end{cases}$$