Original

P4820 Assignment IV

Due, April 6 (negotiable!)

1) A solid sphere of radius a is immersed ina vat of fluid at temperature T_o. Heat is conducted into the sphere according to equation

$$\frac{\partial T}{\partial t} = \frac{kA}{mc}\nabla^2 T = D\nabla^2 T$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

If the temperature of the boundary is fixed at T_0, and the initial temperature of the sphere is T_1, find the temperature within the sphere as a function of time. Plot the temperature $(T - T_0)/(T_1 - T_0)$ as a function of radius r for $Dt/a^2 = 1/20$, $1/10$, and $1/5$.

Hint: The equation in r is "easier" to deal with if you make the change:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = r\frac{\partial^2}{\partial r^2}(rR).$$

2) Consider a sphere of radius a for which the top half is set to voltage V and the bottom half is set to voltage 0 (grounded). Find the potential for this sphere for values of $r > a$ (ie. the potential on the outside of the sphere). You will recall that we did this problem solving for the potential on the inside of the sphere in class. Create a plot of the potential (contour?) to demonstrate that your solution makes sense, ie. voltages are appropriate at the surface of the sphere and they go to zero at infinity.

3) A cylinder of height h and radius a has the top and bottom grounded. The potential on the wall at $\rho = a$ is V_0. Find the potential inside the cylinder. Plot $\Phi(\rho, z = h/2)/V_0$ for $0 < \rho < a$ and $\Phi(\rho = a/2, z)/V_0$, for $0 < z < h$.

1) Assignment 4 <u>Answer</u>

$$\frac{\partial T}{\partial t} = D \nabla^2 T$$

First use separation to get t out

$T = \tau(R)$

∇^2 in spherical ..

$$R \frac{\partial \tau}{\partial t} = D \cdot \overbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \tau R}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial R\tau}{\partial \theta}\right) + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2 R\tau}{\partial \phi^2}}$$

$$R \frac{\partial \tau}{\partial t} = D \cdot \frac{\tau}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} + \underbrace{0 + 0}_{\text{no angular dependence}}$$

$$\frac{1}{\tau}\frac{\partial \tau}{\partial t} = \frac{D}{R} \cdot \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} = k^2 \leftarrow \text{separation constant}$$

$$= \frac{D}{rR} \cdot \frac{1}{r^{\cancel{2}}} \cdot \cancel{r} \frac{\partial^2 rR}{\partial r^2}$$

$$\frac{1}{\tau}\frac{\partial \tau}{\partial t} = \frac{D}{rR}\frac{\partial^2 rR}{\partial r^2} = k^2$$

For time

$$\frac{1}{\tau}\frac{\partial \tau}{\partial t} = k^2$$

$$\ln \tau = k^2 t$$

$$\tau = e^{-k^2 t}$$

For range

otherwise could not match initial condition!

$$\frac{D}{rR}\frac{\partial^2 rR}{\partial r^2} = -k^2$$

$$\frac{\partial^2 rR}{\partial r^2} = \frac{-k^2}{D}(rR)$$

$$rR = A \sin \frac{kr}{D^{1/2}} + B \cos \frac{kr}{D^{1/2}}$$

~~This problem is eased if I form in terms of θ that $T_0 + T = \Pi$ so that as $t \to \infty$ $\Pi \to T_0$~~

~~For time problem~~

Form problem as

$$T - T_0 = \tau \cdot R$$

$$T = \tau \cdot R + T_0$$

and as $t \to \infty$ $\quad \tau R \to 0$

~~also $T = T_0$ at $r = a$~~

also for $r \to a$ $\quad T = T_0$

$$T - T_0 = e^{-k^2 t} \cdot \frac{1}{r}\left[A \sin \frac{k}{D^{1/2}} r + B \cos \frac{k}{D^{1/2}} r\right]$$

~~BLM~~.. For r, need $R(r)\Big|_{r=a} = 0$

set $B = 0$ why and require $\frac{k}{D^{1/2}} a = n\pi$

$$k = \frac{n\pi D^{1/2}}{a}$$

general solution is

$$T - T_0 = e^{-\left(\frac{n\pi}{a}\right)^2 D t} \cdot A \frac{1}{r} \sin \frac{n\pi D^{1/2}}{a} r$$

$$\left.\frac{\cos ar}{r}\right|_{r=0} = \infty \leftarrow$$ B solutions are unbounded at $r=0$!

$$\left.\frac{\sin ar}{r}\right|_{r=0} = 1$$

$$T - T_0 = A_n e^{-\left(\frac{n\pi}{a}\right)^2 Dt} \cdot \frac{1}{r} \sin \frac{n\pi \cancel{D^{1/2}}}{a \cancel{D^{1/2}}} r$$

at $t=0$, $T - T_0 = T_1 \quad \forall r$

$T = T_1$

$$\sum_{n=0}^{\infty} A_n \frac{1}{r} \sin \frac{n\pi}{a} r = (T_1 - T_0) \quad r \leq a$$

$$\sum_{n=0}^{\infty} \int_0^a A_n \sin \frac{n\pi}{a} \cdot \sin \frac{m\pi}{a} \, da =$$

$$\int_0^a (T_1 - T_0) \, r \sin \frac{m\pi}{a} \, dr$$

$$\frac{A_n \cdot a}{2} = \int_0^a (T_1 - T_0) \overset{u}{r} \overset{dv}{\sin \frac{n\pi}{a} r \, dr}$$

$$-(T_1 - T_0) \, r \frac{a}{n\pi} \cos \frac{n\pi}{a} r \Big|_0^a$$

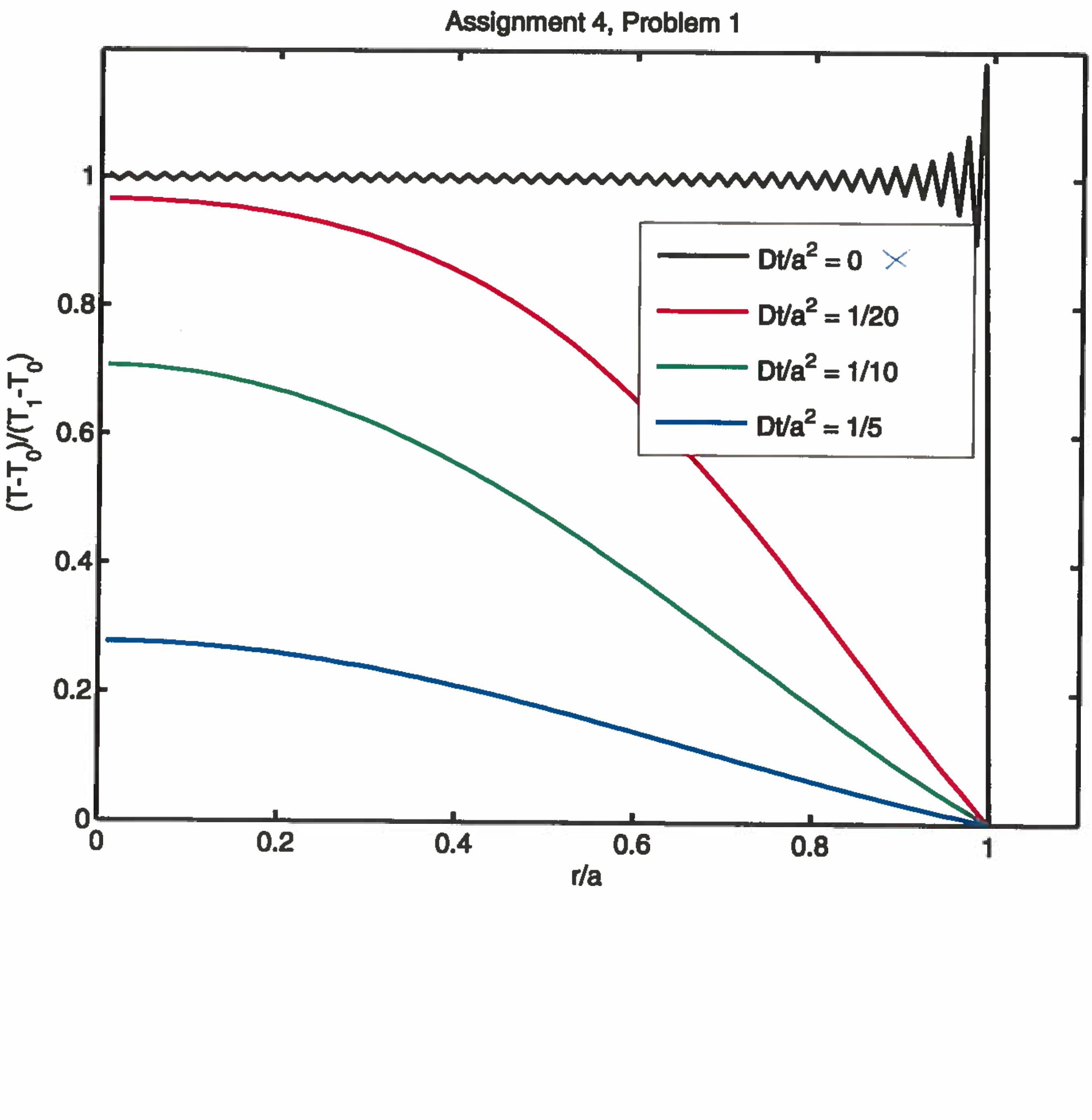
$$-(T_1 - T_0) \int_0^a \frac{-a}{n\pi} \cos \frac{n\pi}{a} r \, dr$$

$$= -(T_1 - T_0) \, a^2/n\pi \cos n\pi + (T_1 - T_0) \frac{+a}{n\pi} \cdot \frac{a}{n\pi} \sin \frac{n\pi}{a} r \Big|_0$$

$$-(T_1 - T_0) \frac{a^2}{n\pi} (-1)^n + (T_1 - T_0) \frac{a^2}{n^2\pi^2} \cancel{\sin n\pi}$$

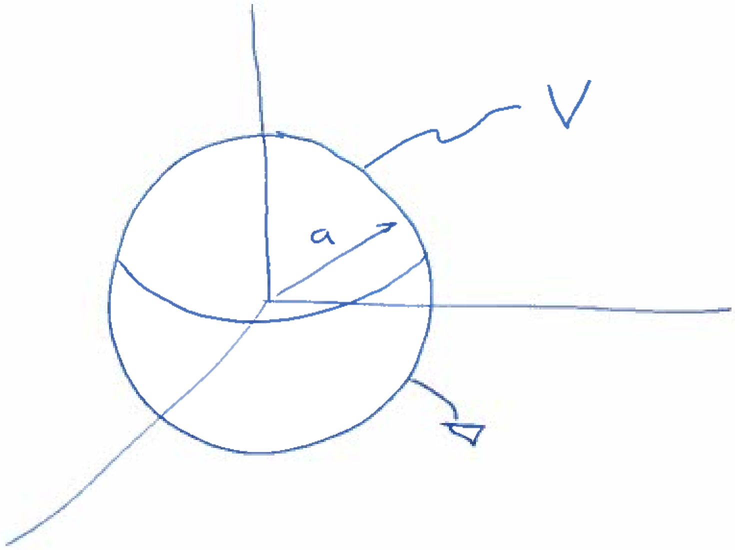
$$A_n = \frac{-2a}{n\pi} (-1)^n \, (T_1 - T_0)$$

$$\therefore \frac{(T-T_0)}{(T_1-T_0)} = \sum \frac{-2\cancel{(T_1-T_0)}}{n\pi}(-1)^n \cdot e^{-\left(\frac{n\pi}{a}\right)^2 Dt} \cdot \frac{1}{r}\sin\frac{n\pi}{a}r$$



Assignment 4, Problem 1
$(T-T_0)/(T_1-T_0)$
r/a
0
0.2
0.4
0.6
0.8
1
$Dt/a^2 = 0$
$Dt/a^2 = 1/20$
$Dt/a^2 = 1/10$
$Dt/a^2 = 1/5$

Ass 4, 2



Hollow conducting sphere, radius a bottom half grounded, top half set to V. Find potential ~~fro~~ for $r > a$.

$\nabla^2 \Phi = 0$ and system has axisymmetry about z axis.

Legendre Polynomials

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\mu) \qquad (1)$$

notice that as $r \to \infty$ the $A_l r^l \to \infty$ NOT GOOD! We avoid this by setting $\boxed{A_l = 0}$

Now, find B by evaluating (1) at $r = a$ for known b.c. value ....

$$\Phi(a,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\mu) = \begin{cases} V & 1 \geq \mu > 0 \\ 0 & 0 > \mu \geq -1 \end{cases}$$

Use orthogonality

$$\int_{-1}^{1} \sum_{\ell=0}^{\infty} \frac{B_\ell}{a^{\ell+1}} P_\ell(\mu) P_{\ell'}(\mu)\, d\mu = \int_{-1}^{1} V(\mu) P_{\ell'}(\mu)\, d\mu$$

$$\frac{B_\ell}{a^{\ell+1}} \underbrace{\int_{-1}^{1} P_\ell(\mu) P_\ell(\mu)\, d\mu}_{\frac{2}{2\ell+1}} = V \int_{0}^{1} P_\ell(\mu)\, d\mu$$

← orthogonality condition

$$\frac{B_\ell}{a^{\ell+1}} \cdot \frac{2}{2\ell+1} = V \underbrace{\int_{0}^{1} P_\ell(\mu)\, d\mu}$$

integrate using step up/down properties ... see notes (my page 99)

$$= \frac{-V\, P_{\ell+1}(0)}{\ell}$$

$$\boxed{B_\ell = \frac{(2\ell+1)\, a^{\ell+1}}{2} \cdot \frac{-V\, P_{\ell+1}(0)}{\ell}}$$

$\ell = 0$ is special case → $\emptyset$

For $\ell = 0$ $P_0(\mu) = 1$

$$\int_0^1 P_0(\mu)\, d\mu = \int_0^1 1\, d\mu = 1$$

So

$$\frac{2B_0}{a} = V\int_0^1 P_0(\mu)\, d\mu = V$$

$$\boxed{B_0 = \frac{aV}{2}}$$

$$\Phi(r,\theta) = \frac{aV}{2r}\left[1 - \frac{2\ell+1}{\ell}\frac{a^\ell}{r^\ell} P_{\ell+1}(0)\cdot P_\ell(\mu)\right]$$

Watch out, must remember to multiply B_0 by $\frac{1}{r^{\ell+1}} = \frac{1}{r}$!

Ass 4, 2

```
clear
close all
a = 1;
V = 1;

step = .05;
x = (-1:step:1)'* ones(1,length(-1:step:1));
y = x';

r = sqrt(x.^2+y.^2);

highs = find(r > 1);
r(highs) = NaN;

theta = atan2(y,x);
theta(highs) + NaN;

maxl = 20;

ttest = 0;
if (ttest == 0)

for ix = 1:length(x)
    for iy = 1:length(y)
        phi(ix,iy) = 0;
        rr = r(ix,iy);
        tt = theta(ix,iy);

        work = 0;
        for l = 1:maxl
            Plo = legendre(l+1,0);
            Plu = legendre(l,cos(tt));
%           work = work + (2*l)/l * Plo(1) * (rr/a).^l * Plu(1);
          work = work + (2*l+1)/l * Plo(1) * (rr/a).^l * Plu(1);
        end
        phi(ix,iy) = V/2*(1-work);
    end
end


figure(1)
clf

imagesc(-1:step:1,-1:step:1,phi)
set(gca,'YDir','normal')
colorbar

figure(2)
clf
contour(-1:step:1,-1:step:1,phi)
set(gca,'YDir','normal')
```

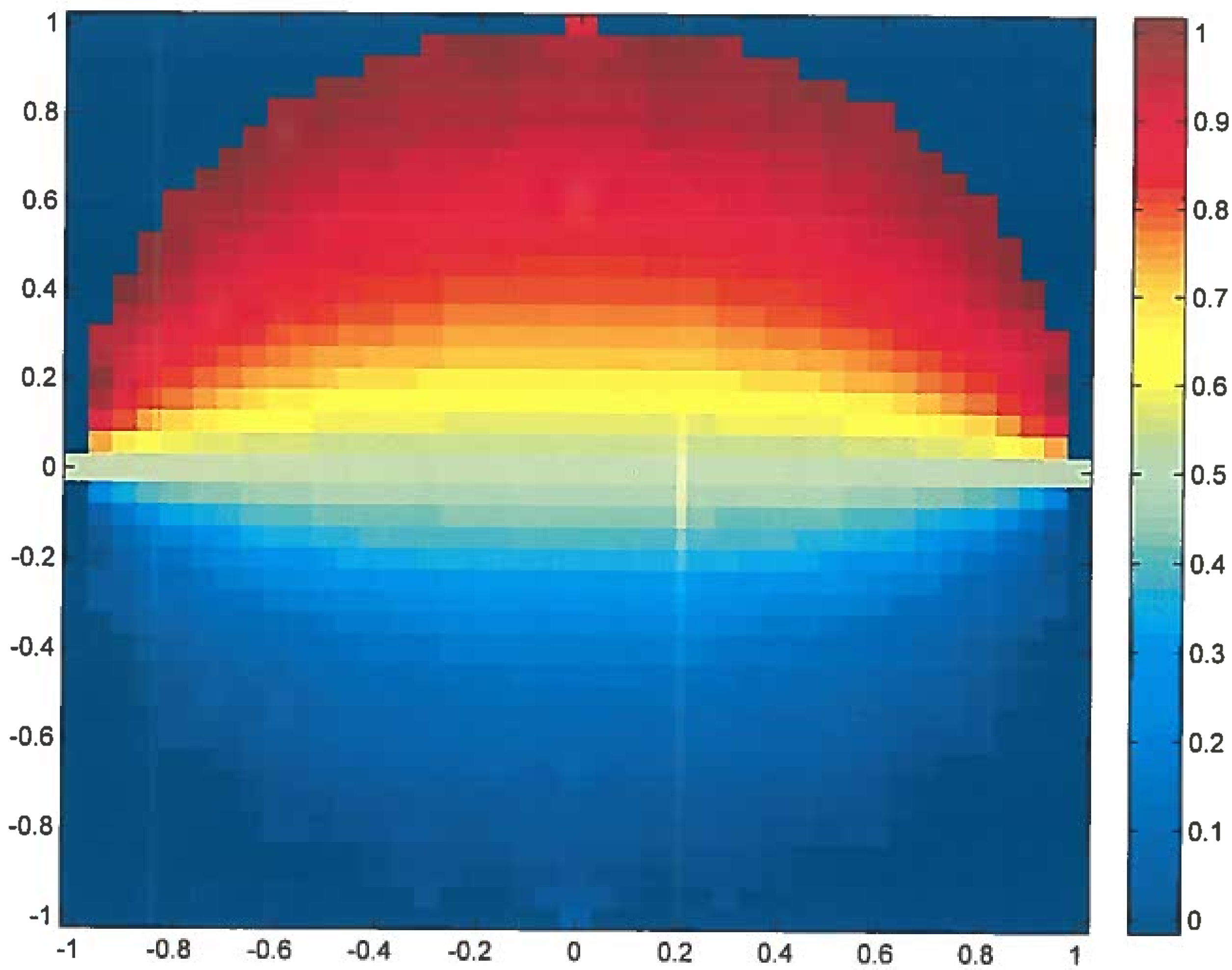
Inside

```
colorbar
end

% Now do r > a!
```

Outside

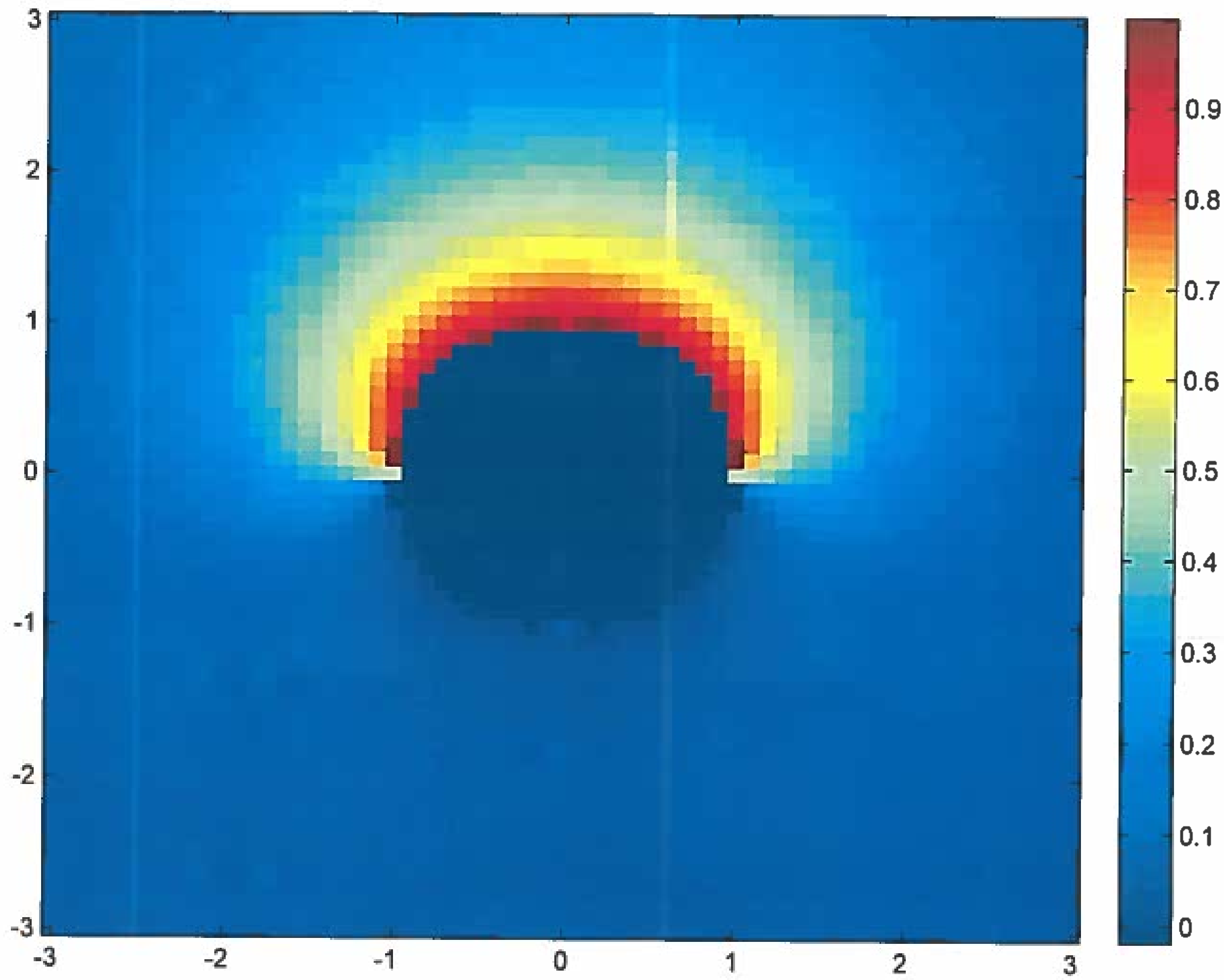
```
step = .1;
mxval = 3;
x = (-mxval:step:mxval)'* ones(1,length(-mxval:step:mxval));
y = x';

r = sqrt(x.^2+y.^2);

lows = find(r < 1);
r(lows) = NaN;

theta = atan2(y,x);
theta(lows) + NaN;

maxl = 20;

for ix = 1:length(x)
    for iy = 1:length(y)
        phi(ix,iy) = 0;
        rr = r(ix,iy);
        tt = theta(ix,iy);

        work = 0;
        for l = 1:maxl
            Plo = legendre(l+1,0);
            Plu = legendre(l,cos(tt));
%             work = work + (2*l)/l * Plo(1) * (a/rr).^(l+1) * (2*l + 1)/l * Plu(1);
            work = work + (2*l+1)/l * Plo(1) * a.^l * (1/rr).^(l) * Plu(1);
        end
        phi(ix,iy) = a*V/2/rr*(1-work);
        %phi(ix,iy) = a*V/2/rr*(1-work);
    end
end

figure(3)
clf

imagesc(-mxval:step:mxval,-mxval:step:mxval,phi)
set(gca,'YDir','normal')
colorbar

figure(4)
clf
contour(-mxval:step:mxval,-mxval:step:mxval,phi)
set(gca,'YDir','normal')
colorbar
```

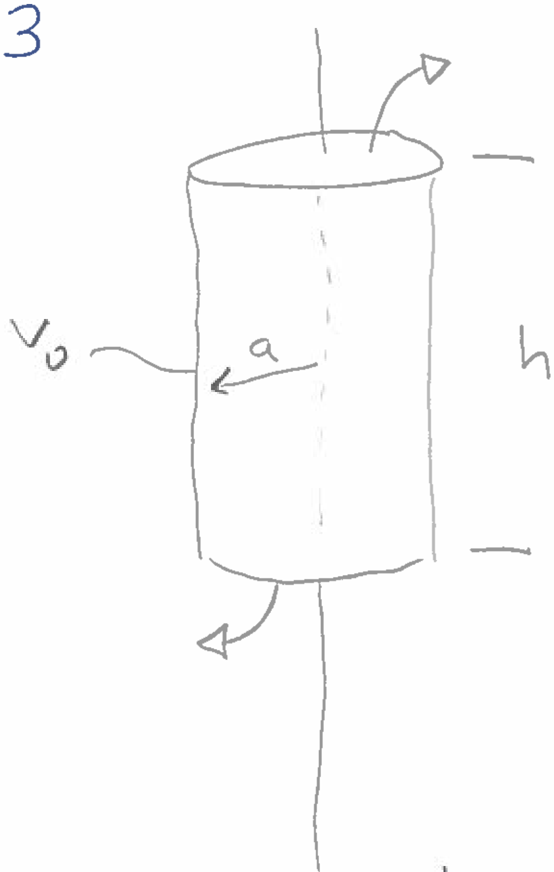
Inside



1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

Outside





Want potential inside.

--- $\nabla^2 \Phi = 0$

This problem has axial symmetry so

$$W = e^{\pm i m \phi}$$

has $\underline{m=0}$.

General solution for $Z = e^{ikz}$

need sin/cos form to match 0 at $z=0$ and $z=h$.

This form in z forces $-$ sign in radial (ρ) eqn and you get solutions I_n and K_n. K_n's cannot be solutions because they are not finite at $\rho = 0$. The only non-zero I_n is $\underline{\underline{I_0}}$

General Form Solution:

$$\Phi(\rho, z) = \sum A_n \sin\left(\frac{n\pi z}{h}\right) I_0\left(\frac{n\pi}{h}\rho\right)$$

Set $\rho = a$ and find $\Phi(a, z) = V_0$

2

$$V_0 = \sum_n A_n \sin\left(\frac{n\pi z}{h}\right) I_0\left(\frac{n\pi}{h} a\right)$$

$$\frac{V_0}{I_0\left(\frac{n\pi}{h} a\right)} = \sum A_n \sin \frac{n\pi z}{h}$$

$$\int_0^h \frac{V_0}{I_0\left(\frac{n\pi}{h} a\right)} \cdot \sin\left(\frac{n\pi}{h} z\right) dz = A_n \int_0^h \sin^2\left(\frac{n\pi}{h} z\right) dz$$

$$\frac{V_0}{I_0\left(\frac{n\pi}{h} a\right)} \underbrace{\int_0^h \sin \frac{n\pi}{h} z \; dz}_{} = A_n \frac{h}{2}$$

$$\left[-\cos\left(\frac{n\pi}{h} z\right) \cdot \frac{h}{n\pi}\right]_0^h = -\cos n\pi - -1$$

$$= \frac{h}{n\pi}\left(1 - \cos n\pi\right)$$

$$A_n \frac{h}{2} = \frac{V_0}{I_0\left(\frac{n\pi}{h}\right) a} \cdot \frac{h}{n\pi}\left(1 - \cos n\pi\right)$$

$\cos n\pi = -1$ n odd

$\cos n\pi = 1$ n even

$$\boxed{A_n = \frac{V_0}{I_0\left(\frac{n\pi}{h} a\right)} \cdot \frac{4}{n\pi}}$$

$$\Phi(\rho, z) = \sum_{\substack{n \\ \text{odd}}} \frac{V_0}{I_0\left(\frac{n\pi}{h} a\right)} \cdot \frac{4 \sin\left(\frac{n\pi}{h} z\right)}{n\pi} \cdot \frac{I_0\left(\frac{n\pi}{h} \rho\right)}{I_0\left(\frac{n\pi}{h} a\right)}$$

$$\Phi(\rho,z) = \sum_n A_n \sin\left(\frac{n\pi z}{h}\right) I_0\left(\frac{n\pi}{h}\rho\right)$$

Now evaluate the potential at $\rho = a$:

$$V_0 = \sum_n A_n \sin\left(\frac{n\pi z}{h}\right) I_0\left(\frac{n\pi}{h}a\right)$$

This is a Fourier sine series, and we find the coefficients in the usual way:

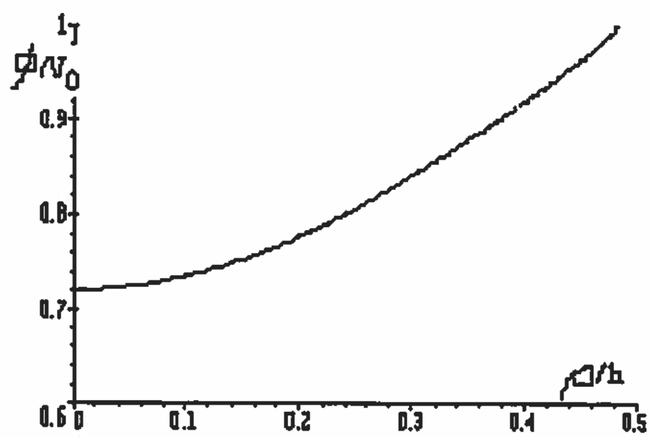
$$A_n \frac{h}{2} = \frac{V_0}{I_0\left(\frac{n\pi}{h}a\right)} \int_0^h \sin\left(\frac{n\pi z}{h}\right) dz = \frac{V_0}{I_0\left(\frac{n\pi}{h}a\right)} \frac{h}{n\pi}(1 - \cos n\pi)$$

$$= \frac{V_0}{I_0\left(\frac{n\pi}{h}a\right)} \frac{2h}{n\pi} \text{ for } n \text{ odd}$$

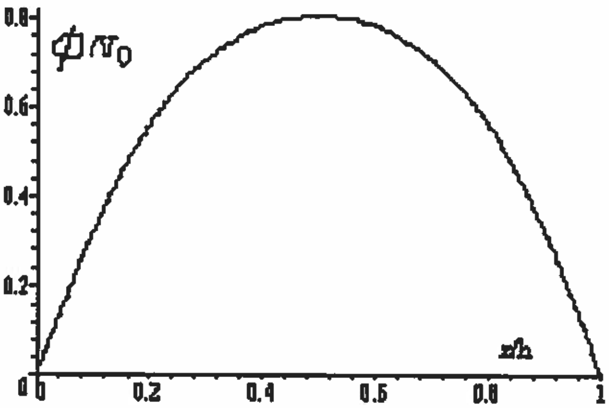
Thus

$$\Phi(\rho,z) = \frac{4V_0}{\pi} \sum_{n \text{ odd}} \frac{\sin\left(\frac{n\pi z}{h}\right)}{n} \frac{I_0\left(\frac{n\pi}{h}\rho\right)}{I_0\left(\frac{n\pi}{h}a\right)}$$

The first plot shows the first 9 terms in the expansion of $\Phi(\rho, h/2)/V_0$ for $a = h/2$.



The second plot shows $\Phi(a/2,z)/V_0$ with $a = h/2$.



43. A cylinder of height h and radius a is grounded except for its base at $z = 0$. On the base the potential is $\dfrac{V_0}{\sqrt{1-\rho^2/a^2}}$. Find the potential inside the cylinder.

Again we have axisymmetry, and we must choose the eigenvalue to make the potential zero at $\rho = a$. To make the potential zero at $\rho = h$ we use the hyperbolic sine function.

$$\Phi(\rho,z) = \sum_n A_n \sinh\left(x_{0n}\frac{(h-z)}{a}\right)J_0\left(x_{0n}\frac{\rho}{a}\right)$$

where

$$A_n \sinh\left(\frac{h}{a}x_{0n}\right)\frac{a^2}{2}\left[J_0'(x_{0n})\right]^2 = \int_0^a \frac{V_0}{\sqrt{1-\rho^2/a^2}}J_0\left(x_{0n}\frac{\rho}{a}\right)\rho d\rho$$

$$= a^2\int_0^1 \frac{V_0}{\sqrt{1-y^2}}J_0(x_{0n}y)ydy$$

$$= V_0 a^2\frac{\sin x_{0n}}{x_{0n}}$$

We used GR 6.554#2 to evaluate the integral. . Thus

$$\sum_n 2V_0\frac{\sin x_{0n}}{x_{0n}}\frac{\sinh\left(x_{0n}\frac{(h-z)}{a}\right)}{\sinh\left(\frac{h}{a}x_{0n}\right)}\frac{J_0\left(x_{0n}\frac{\rho}{a}\right)}{\left[J_0'(x_{0n})\right]^2}$$

44. (a) Use the series for $J_0(x)$ to show that its Laplace transform is