P4820 Assignment III

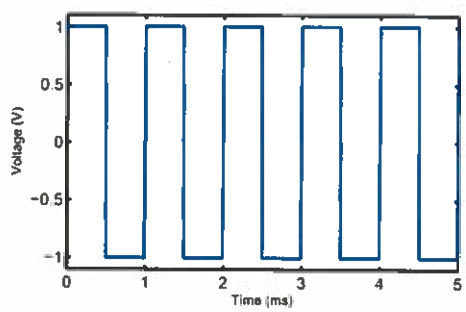
Due, March 8, 2019

1) [5] What distribution is represented by the relation $e^{-x}\,\delta'(x)$?

2a) [5] Find the current as a function of time for an RC circuit driven by a square wave from a function generator as shown below. Kirchoff's loop rule gives:

$$V(t) = q(t)/C + i(t)R \tag{1}$$

where $V(t)$ is the forcing voltage, $q(t)$ is the charge on capacitance C, R is the resistance, and $i(t)$ is the current. Express the CURRENT in the circuit in terms of delta functions, and solve for the current as a function of time. You will need to use the Fourier series representation of the delta function which conveniently repeats periodically. Plot the result for a circuit with R = 100 Ω, C = 2.0×10^{-5} F, and T = 0.001 s;



2b) [10] (Forgive me but the result is a teachable moment) Repeat the circuit analysis using Laplace transforms (just to keep in practice). You can't assume that the initial current is 0 so you need to figure out what it should be by requiring the time average current to be 0!

2c) [5] Graph your results from parts 2a and 2b and compare the two solutions.

3) [10] In class, we considered the differential equation:

$$m\frac{dv}{dt} + \alpha v = F(t) \tag{2}$$

which represents the velocity of an object acted on by some force $F(t)$ where there is a drag term (represented by the αv). and found the Green's function to be:

$$G(t,t') = \begin{cases} 0 & \text{if} \quad t < t' \\ Ae^{-\alpha t/m} & \text{if} \quad t > t' \end{cases}. \tag{3}$$

Find the velocity of the object that results if the forcing term is given as:

$$F(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ \beta t & \text{if} \quad 0 < t < T \\ 0 & \text{if} \quad t > T \end{cases} \tag{4}$$

Plot the velocity as a function of time given that $\alpha = 1$ Ns/m, $\beta = 1$ N/s, $m = 1$ kg, and $T = 1$. (plot the graph out to t/T = 5).

(1)

$$\int e^{-x}\,\delta'(x)\,g(x)\,dx$$

$$= \int \delta'(x)\,e^{-x}\,g(x)\,dx$$

$$= -\frac{d}{dx}\left[e^{-x}\,g(x)\right]\Big|_0$$

$$= e^{-x}g(x) - e^{-x}g'(x)\Big|_0$$

$$= g(0) - g'(0)$$

$$\rightarrow \quad e^{-x}\,\delta'(x) = \delta(x) + \delta'(x) \quad !$$

Can also think as

must use δ sequence, otherwise what does * this term equal!

$$\int e^{-x}\,\delta'(x)\,g(x)\,dx = \lim_{n\to\infty} \int e^{-x}\,\phi_n'(x)\,g(x)\,dx$$

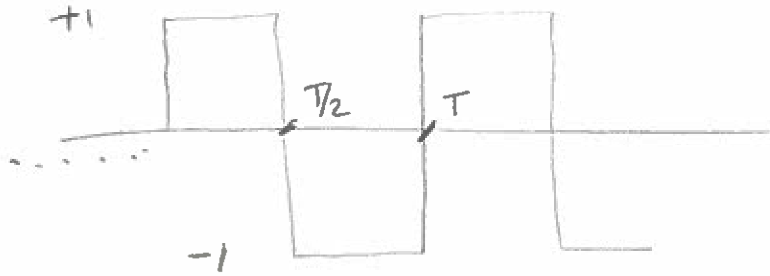
key point.

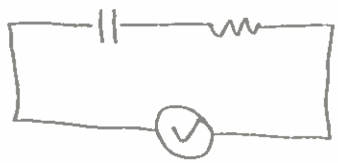
$$= \lim_{n\to\infty}\int \phi_n'(x)\,e^{-x}g(x)\,dx = \lim_{n\to\infty} \underbrace{\phi_n(x)\,e^{-x}\,g(x)\Big|_{-\infty}^{\infty}}_{0} - \lim_{n\to\infty}\int \phi_n(x)\left[-e^{-x}g(x) - e^{-x}g'(x)\right]$$

$$= \lim_{n\to\infty}\int \phi_n(x)\,e^{-x}g(x)\,dx - \lim_{n\to\infty}\int \phi_n(x)\,e^{-x}g'(x)\,dx$$

$$= g(0) - g'(0) \quad \text{as before}$$

2) RC circuit driven by square wave





In this circuit,

$$V = \frac{Q}{C} + RI$$

d/dt of eqn

$$\boxed{\frac{dV}{dt} = \frac{1}{C} I + R\frac{dI}{dt}}$$

Consider input as sequence of step functions

$$V(t) = \left[-1 + 2S(t) - 2S\left(t-\tfrac{T}{2}\right) + 2S(t-T) - \dots\right] V_0$$

So that

$V_0 = 1$! in this case

$$\frac{dV}{dt} = 2V_0\delta(t) - 2V_0\delta\left(t-\tfrac{T}{2}\right) + 2V_0\delta(t-T) + \dots$$

and this can be represented by the series representation of the δ function →

$$\delta = \frac{1}{T} \sum e^{2m\pi i\, t/T}$$

(notice that this repeats with period T)

$$\frac{dV(t)}{dt} = 2V_0/T \sum e^{2\pi m i\, t/T} - 2V_0/T \sum e^{2\pi m i \frac{(t - T/2)}{T}}$$

$$= \frac{2V_0}{T} \sum e^{2\pi i m\, t/T} - \frac{2V_0}{T} \sum e^{2\pi i m\, t/T} \cdot \underbrace{e^{-2\pi m i/2}}$$

$$e^{-m\pi i} = \begin{cases} 0 & m = 0, \pm 2, \pm 4 \\ -1 & m = \pm 1, \pm 2, \ldots \end{cases}$$

$$\boxed{\frac{dV}{dt} = \frac{4V_0}{T} \sum_{n\ \mathrm{odd}} e^{2\pi i m\, t/T}}$$

Now, let $I = \sum_n I_n e^{in2\pi\, t/T}$ — assumes a Fourier form for I → match form of dV/dt

$$\frac{4V_0}{T} \sum_{n\mathrm{odd}} e^{2\pi i m\, t/T} = R \sum_n \frac{2\pi i n}{T} I_n e^{2\pi i n t/T} + \frac{1}{C} \sum I_n e^{2\pi i n t/T}$$

for odd n,

$$\frac{4V_0}{T} = \left(\frac{2\pi i R}{T} n + \frac{1}{C} \right) I_n$$

for even n

$I_n = 0$!

$$I_n = \frac{4V_0}{T} \cdot \frac{1}{\frac{2\pi i R}{T} n + \frac{1}{c}}$$

$$\boxed{I_n = \frac{4V_0}{2\pi i R n + T/c}}$$

$$\boxed{I(t) = \sum_{n \text{ odd}} \frac{4V_0}{2\pi i R n + T/c} \cdot e^{2\pi i n t/T}}$$

Now using Laplace transforms
Same equation...

$$\frac{dv}{dt} = \Big[2V\,\delta(t) - 2V\,\delta\!\left(t - \tfrac{T}{2}\right)\Big]$$ repeating with period T

Take transform on initial $0 \to T$ and then use the repeating property of Laplace

$$\int_0^{\infty} e^{-st} \cdot 2V\,\delta t = 2V$$

$$-\int_0^{\infty} e^{-st}\, 2V\,\delta(t - T/2) = -2V e^{-sT/2} \quad s$$

$$\frac{dV}{dt}(s) = \frac{2V(1 - e^{-sT/2})}{1 - e^{-sT}}$$

add denominator for repeating function with period T!

Take Laplace Transform of eqns

$$\frac{2V\left(1-e^{-sT/2}\right)}{1-e^{-sT}} = sR\,I(s) + \frac{1}{C}I(s) - R\,i_0$$

initial current.

$$I = \frac{2V}{sR+1/C}\cdot\frac{1-e^{-sT/2}}{1-e^{-sT}} + \frac{R\,i_0}{sR+1/C}$$

$$= \frac{2V/R}{s+1/RC}\cdot\frac{1-e^{-sT/2}}{1-e^{-sT}} + \frac{i_0}{s+1/RC}$$

$$= \left[\frac{2V}{R}\,\frac{1}{s+1/RC} - \frac{2V}{R}\,e^{-sT/2}\cdot\frac{1}{s+1/RC}\right]\cdot\frac{1}{1-e^{-sT}}$$

$$+\; i_0\left[\frac{1}{s+1/RC}\right]$$

take inverse transform

$$\mathcal{L}^{-1}\left[\frac{1}{s+1/RC}\right] = e^{-t/RC}$$

also notice time shift term

$$I(t) = \underbrace{\left[\frac{2V}{R}\,e^{-t/RC} - \frac{2V}{R}\,e^{-(t-T/2)/RC}\,S\!\left(t-\tfrac{T}{2}\right)\right]}_{\text{repeating with period } T} + i_0\,e^{-t/RC}$$

To find i_0, assume that mean current must be 0

$$\frac{1}{T}\int_0^T I\,dt = 0$$

also notice that you can pick $t=0$ anywhere as the whole thing repeats periodically.

$$I(t) = \left[\frac{2V}{R}e^{-t/RC} - \frac{2V}{R}e^{-(t-T/2)/RC}\,S(t-T/2)\right] + i_0 e^{-t/RC}$$

(a) (b) repeating T period (c)

(a) $$\frac{1}{T}\int_0^T \frac{2V}{R}e^{-t/RC}\,dt = \frac{-2VC}{T}e^{-t/RC}\Big|_0^T = \frac{2VC}{T} - \frac{2VC}{T}e^{-T/RC}$$

(b) $$\frac{1}{T}\int_{T/2}^T -\frac{2V}{R}e^{-(t-T/2)/RC}\,dt = \frac{2VC}{T}e^{-(t-T/2)/RC}\Big|_{T/2}^T$$

Step function

$$= \frac{2VC}{T}e^{-T/2RC} - \frac{2VC}{T}$$

(c) Notice i_0 will be the same every cycle so that

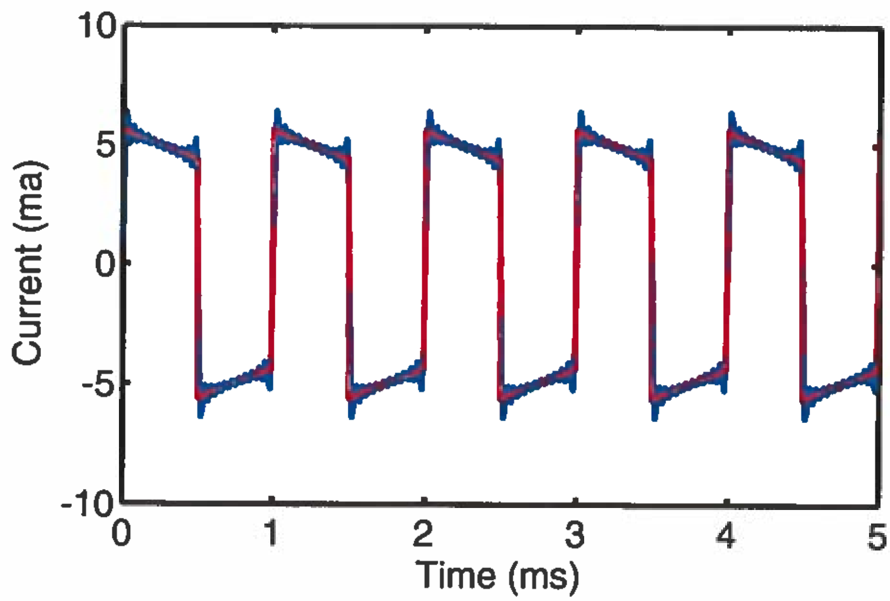
$$\frac{1}{T}\int_0^T i_0 e^{-t/RC} = -\frac{i_0 RC}{T}e^{-T/RC} + \frac{RCi_0}{T}$$

$a + b + c$ must $= 0$

$$\overset{c}{\frac{RC\, i_0}{T}\left(1 - e^{-T/RC}\right)} + \overset{b}{\frac{2VC}{T}\left(e^{-T/2RC} - 1\right)} + \frac{2VC}{T}\left(1 - e^{-T/RC}\right) = 0$$

$$i_0 = \frac{2VC\left[e^{-T/RC} - 1 + 1 - e^{-T/2RC}\right]}{1 - e^{-T/RC}}$$

$$i_0 = \frac{2VC}{RC}\left[\frac{e^{-1/2} - 1 + 1 - e^{-1/4}}{1 - e^{-1/2}}\right] = -8.7564\times10^{-3}$$



Red is Laplace Solution
Blue is series solution

- Critical difference is that the Laplace solution is exact while the series solution is an approximation

Q3 2011

$$F(t) = \begin{cases} 0 & t<0 \\ \beta t & t>0, <T \\ 0 & t>T \end{cases} \qquad \beta \equiv \frac{N}{t} \qquad \alpha = \frac{N}{m/s}$$

$$v(t) = \int_{-\infty}^{\infty} F(t') \, G(t,t') \, dt' = \int_{-\infty}^{\infty} \begin{Bmatrix} 0 & t'<0 \\ \beta t' & 0<t'<T \\ 0 & t'>T \end{Bmatrix} \cdot G(t,t') dt'$$

As before, $v(t) \quad t<0 = 0$

for $t>0$

$$v(t) = \int_0^T \beta t' \, G(t,t') \, dt'$$

again for $t<T$

$$v(t) = \int_0^t \beta t' \, G(t,t') \, dt'$$

$$= \int_0^t \beta t' \frac{1}{m} e^{-\frac{\alpha}{m}(t-t')} dt' \qquad \frac{kg \frac{m}{s^3}}{kg} s^2$$

$$= \frac{\beta}{m} e^{-\frac{\alpha t}{m}} \underbrace{\int_0^t \underset{v}{t'} \underset{du}{e^{+\frac{\alpha}{m} t'}} dt'} \qquad \frac{kg \frac{m}{s^2}}{kg} \cdot s$$

$$= \frac{\beta}{m} e^{-\frac{\alpha t}{m}} \left[\underset{v}{t'} \underset{u}{\frac{m}{\alpha} e^{\frac{\alpha}{m} t'}} \Big|_0^t - \int_0^t \frac{m}{\alpha} e^{\frac{\alpha}{m} t'} dt' \right]$$

$$\frac{kg}{kg \frac{m}{s}} \cdot \frac{m}{s}$$

$$= \frac{\beta}{m} e^{-\alpha t/m} \left[\left(t \frac{m}{\alpha} e^{\frac{\alpha}{m} t} - 0 \right) - \frac{m^2}{\alpha^2} e^{\frac{\alpha}{m} t} \Big|_0^t \right]$$

$$= \frac{\beta}{\cancel{m}} e^{-\frac{\alpha t}{m}} \left[\frac{\cancel{m}}{\alpha} t e^{\frac{\alpha}{m} t} - \frac{m^{\cancel{2}}}{\alpha^2} e^{\frac{\alpha}{m} t} + \frac{m^{\cancel{2}}}{\alpha^2} \right]$$

$$= \frac{\beta}{\alpha} t - \frac{\beta m}{\alpha^2} + \frac{\beta m}{\alpha^2} e^{-\alpha t/m}$$

$$= \boxed{\frac{\beta}{\alpha} t + \frac{\beta m}{\alpha^2} \left(e^{-\alpha t/m} - 1 \right)} \qquad t < T$$

For $t > T$

$$v(t) = \int_0^T \beta t' \frac{1}{m} e^{-\frac{\alpha}{m}(t - t')} dt'$$

$$= \frac{\beta}{m} e^{-\frac{\alpha t}{m}} \int_0^T t' e^{\frac{\alpha}{m} t'} dt'$$

$$= \frac{\beta}{m} e^{-\frac{\alpha t}{m}} \left[\frac{m}{\alpha} t' e^{\frac{\alpha t'}{m}} \Big|_0^T - \int_0^T \frac{m}{\alpha} e^{\frac{\alpha}{m} t'} dt' \right]$$

$$= \frac{\beta}{m} e^{-\alpha t/m} \left[\frac{m}{\alpha} T e^{\alpha T/m} - \frac{m^2}{\alpha^2} e^{\alpha/m \, t'} \Big|_0^T \right]$$

$$= \boxed{\frac{\beta}{m} e^{-\alpha t/m} \left[\frac{m}{\alpha} T e^{\alpha T/m} - \frac{m^2}{\alpha^2} e^{\frac{\alpha}{m} T} + \frac{m^2}{\alpha^2} \right]} \qquad t > T$$

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clear

T = 1;

t = 0:.01:(5*T);

alp = 1;
bet = 1;
m = 1;

tlessT = find(t < T);

V = zeros(size(t));

V(tlessT) = bet/alp * t(tlessT) + bet*m/alp^2 .* (exp(-alp.*t(tlessT)/m) - 1);

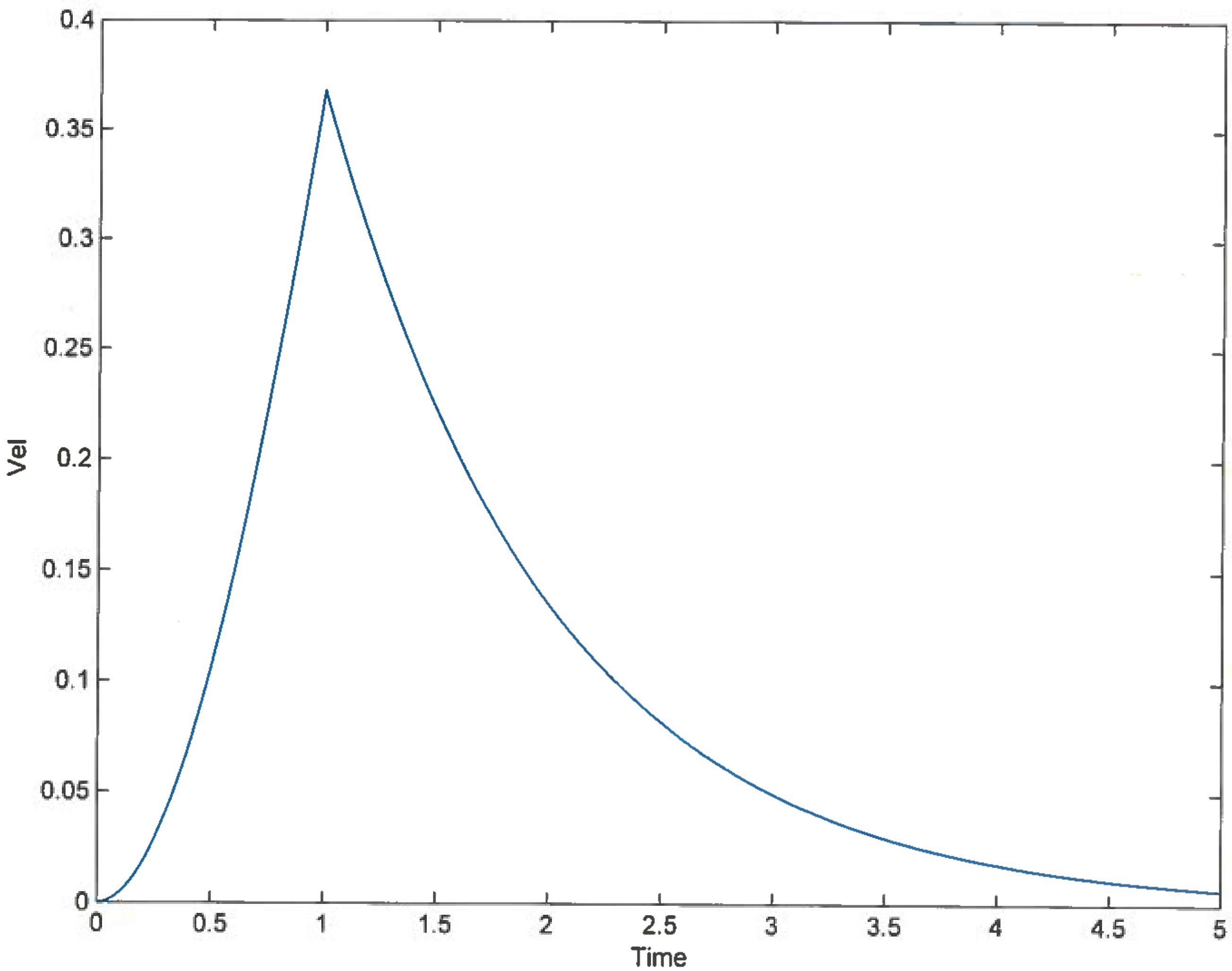
tgreatT = find(t >= T);


V(tgreatT) = bet/m*exp(-alp*t(tgreatT)/m) .* (m/alp * T * exp(alp*T/m) - m^2/alp^2 * exp
(alp/m*T) + m^2/alp^2);

figure(1)
clf
plot(t,V)
xlabel('Time')
ylabel('Vel')


figure(2)
clf
plot(diff(V))
```

as a check did this numerically
in Greens.m



0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Vel
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Time