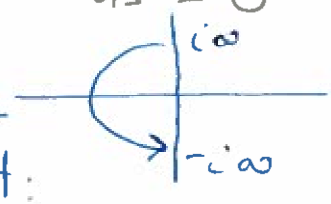
1) Show using Jordan's Lemma that

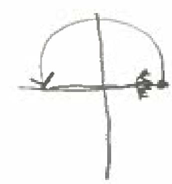
$$\int_{i\infty}^{-i\infty} \frac{3s}{(s+1)(s-3)} e^{st} \, ds = 0 \qquad (1)$$

along the arc ———— 

Jordan's lemma states that:

if $f(z)$ "converges uniformly" to zero when $z \to \infty$ then

$$\lim_{R\to\infty} \int_{C_r} f(z) \, e^{ikz} \, dz = 0 \qquad (2)$$

- k any positive real number
- C_r is the upper half circle $|z| = R$ 

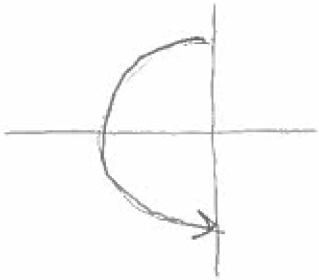
We have to make (1) look like (2) !

$$f(s) = \frac{3s}{(s+1)(s-3)}$$

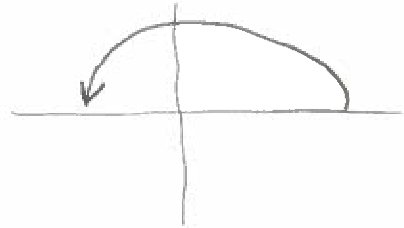
consider (1) if $s = Re^{i\theta}$ then $\int_{i\infty}^{-i\infty}$ can be expressed as

$$\lim_{R\to\infty} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} f(s) \, e^{st} \, dt$$

Things are 90° out of skew. We have



and we need



let $ip = s$

$$\lim_{R\to\infty}\int_0^{\pi} f(ip)\, e^{ipt}\, i\,dp$$

this works so long as

$$\lim_{R\to\infty}\left[f(z) \equiv i\, f(ip) = \frac{i\cdot 3ip}{(ip+1)(ip-3)}\right] \to 0$$

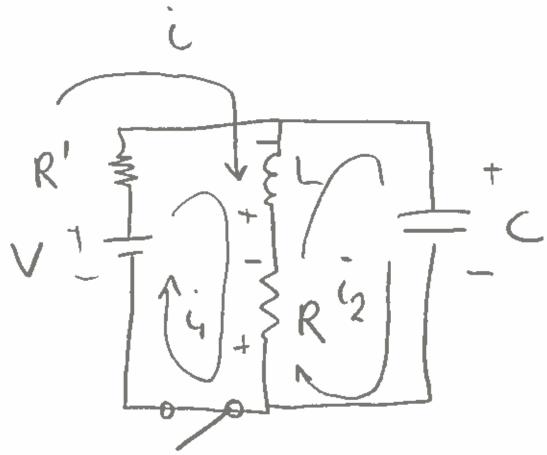
$$\lim_{R\to\infty} \frac{-3p}{-p^2-3ip+ip-3} = \lim_{R\to\infty}\frac{3p}{p^2+2ip+3}$$

$p = Re^{i\theta}$

$$\lim_{R\to\infty}\frac{3Re^{i\theta}}{R^2e^{2i\theta}+\dots} \to \lim_{R\to\infty}\frac{1}{R} \to 0 \;!$$

our integral $\boxed{\int_{i\infty}^{-i\infty}\frac{3s}{(s+1)(s-3)}\, e^{st}\, ds = 0}$

2)



After a long time with the switch closed!

$$i = \frac{V}{R'+R}$$

$$\boxed{V_{c_{initial}} = i \cdot R = \frac{VR}{R'+R}}$$

charge on capacitor is $CV = Q$

$$q_{init} = \frac{CVR}{R'+R}$$

When the switch has been open for a long time, the charge on the capacitor $\to 0$ and the current through the inductor $\to 0$

2 cont

When switch is open, $i_1 = 0$

So I'm left with

$$L\frac{di}{dt} + \frac{q}{C} + iR = 0$$

$$L\frac{d^2i}{dt^2} + \frac{i}{C} + \frac{di}{dt}R = 0$$

$$L\cdot\left[-\cancel{\frac{di}{dt}\Big|_0} - s\,i(0) + s^2 I\right] + \frac{1}{C}I + R\left[-i(0) + sI\right] = 0$$

0

because of L

$L\frac{di}{dt}\Big|_0 = 0$!

call this i_0

$$-Lsi_0 + Ls^2 I + \frac{1}{C}I - Ri_0 + RsI = 0$$

$$\left[Ls^2 + Rs + \frac{1}{C}\right]I = Lsi_0 + Ri_0$$

$$I = \frac{(Ls+R)\,i_0}{Ls^2 + Rs + \frac{1}{C}}$$

$$\frac{Ls\, i_0}{Ls^2+Rs+\frac{1}{C}} + \frac{R\, i_0}{Ls^2+Rs+\frac{1}{C}} = I$$

divide by $1/L$ and combine

$$\frac{\left(s+R/L\right) i_0}{s^2+\frac{R}{L}s+\frac{1}{LC}} = I$$

$R/L = 2\alpha \qquad 1/LC = \omega_0^2$

$$i_0 \frac{(s+2\alpha)}{(s+\alpha)^2+(\omega_0^2-\alpha^2)} = I$$

$$i_0 \frac{s+\alpha}{(s+\alpha)^2+\omega^2} + i_0\alpha \frac{1}{(s+\alpha)^2+\omega^2} = I$$

$$i_0 \frac{s+\alpha}{(s+\alpha)^2+\omega^2} + \frac{i_0\alpha}{\omega} \cdot \frac{\omega}{(s+\alpha)^2+\omega^2} = I$$

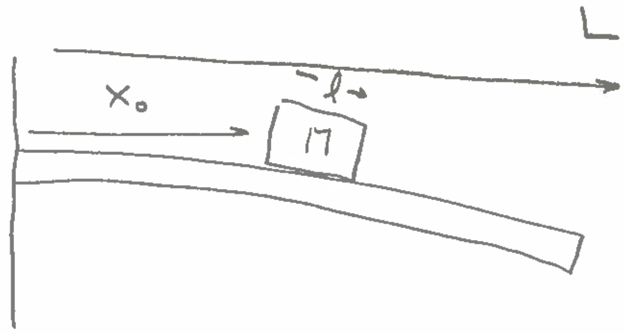
damping term   cos   damping term   sin

$$i_0 e^{-\alpha t} \cos \omega t + \frac{i_0\alpha}{\omega} e^{-\alpha t} \sin \omega t = i(t)$$

3]

$$\frac{d^4 y}{dx^4} = \frac{1}{EI} \cdot q(x)$$

$$q(x) = \begin{cases} \frac{Mg}{\ell} & x_0 < x < x_0 + \ell \\ 0 & \text{otherwise} \end{cases}$$



held horizontaly at $x=0$

at $x=0$, $\boxed{y_0=0}$! $\boxed{y'(0)=0}$

also, since all force is supported at $x=0$,

$$y'''(0) = \frac{-1}{EI} \underbrace{t(x)}_{\text{all supported force}}$$

$$\boxed{y'''(0) = -\frac{1}{EI} Mg}$$

at $x=L$ there can be no supported torque so that $\boxed{y''(L) = 0}$

$$\mathcal{L}\left[\frac{d^4 y}{dx^4}\right] = \mathcal{L}\left[\frac{1}{EI}\, q(x)\right] = \frac{1}{EI}\, Q(s)$$

$$s^4 Y - s^3 \cancelto{0}{y(0)} - s^2 \cancelto{0}{y(0)} - s\, y''(0) - y'''(0) = \frac{1}{EI} Q(s)$$

$$s^4 Y - s\, y''(0) + \frac{Mg}{EI} = \frac{Q(s)}{EI}$$

$$Q(s) = \int_0^\infty q(x)\, e^{-xs}\, dx$$

$$= \int_{x_0}^{x_0+\ell} \frac{Mg}{\ell}\, e^{-xs}\, dx = \frac{Mg}{\ell}\left(\frac{-1}{s}\right) e^{-xs} \Big|_{x_0}^{x_0+\ell}$$

$$= \frac{-Mg}{\ell s}\left[e^{-(x_0+\ell)s} - e^{-x_0 s}\right]$$

$$s^4 Y - s\, y''(0) + \frac{Mg}{EI} = \frac{Mg}{EI\ell s}\left[e^{-x_0 s} - e^{-(x_0+\ell)s}\right]$$

$$Y = \frac{Mg}{EI\ell s^5}\left[e^{-x_0 s} - e^{-(x_0+\ell)s}\right] + \underbrace{\frac{y''(0)}{s^3} - \frac{Mg}{EI s^4}}_{\text{powers of } x}$$

powers of x

translation

$$y(x) = \frac{Mg}{EI\ell}\,\frac{1}{4!}\left[(x-x_0)^4\, S(x-x_0) - \big(x-(x_0+\ell)\big)^4\, S\big(x-(x_0+\ell)\big)\right]$$

$$+\; y''(0)\,\frac{x^2}{2!} \;-\; \frac{Mg}{EI}\,\frac{x^3}{3!}$$

$y'' \propto$ torque to right of $\underrightarrow{x}$ $\quad\rightarrow\quad$ $y''(0) = \dfrac{Mg\cdot\left(x_0+\frac{\ell}{2}\right)}{EI}$

Now, I need to find ~~y(0)~~ $y''(0)$ and I will do that by solving for $y''(L) = 0$

$$y' = \frac{Mg}{EI\ell}\cdot\frac{1}{24}\cdot\left[4\,(x-x_0)^3 - 4\,\big(x-(x_0+\ell)\big)^3\right]$$

$$+\; y''(0)\,\frac{2x}{2} \;-\; \frac{Mg}{EI}\cdot\frac{3x^2}{6}$$

$$y'' = \frac{Mg}{EI\ell}\cdot\frac{1}{24}\cdot\left[12\,(x-x_0)^2 - 12\big(x-(x_0+\ell)\big)^2\right]$$

$$+\; y''(0) \;-\; \frac{Mg}{EI}\cdot\frac{\cancel{6}x}{\cancel{6}}$$

evaluate at L

$$\frac{Mg}{EI}\cdot\frac{1}{2\,\cancel{24}\ell}\left[\cancel{12}\,(L-x_0)^2-\cancel{12}\,(L-(x_0+\ell))^2\right]+y''(0)-\frac{Mg}{EI}L$$

$$\cancel{L^2}-2\cancel{Lx_0}+\cancel{x_0^2}-\left(L^2-2L(x_0+\ell)+(x_0+\ell)^2\right)$$

$$-\left[\cancel{L^2}-2\cancel{Lx_0}-2L\ell+\cancel{x_0^2}+2x_0\ell+\ell^2\right]$$

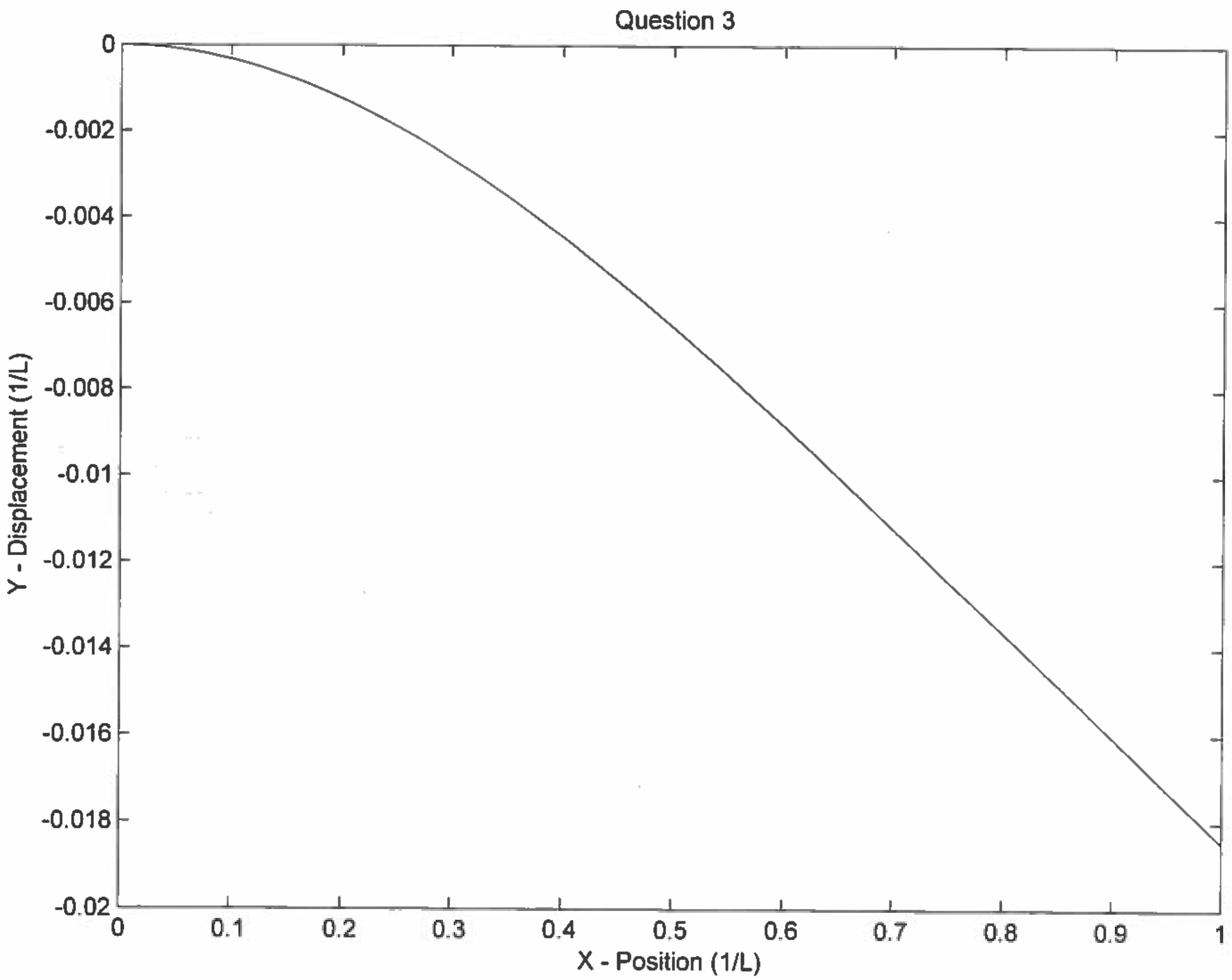
$$y''(L)=\frac{Mg}{EI}\cdot\frac{1}{2\ell}\left(\cancel{+2}L\ell-2x_0\ell-\ell^2\right)-\frac{Mg}{EI}L+y''(0)$$

$$=\frac{Mg}{EI}\cdot\left(\cancel{+L}-x_0-\frac{\ell}{2}-\cancel{L}\right)+y''(0)$$

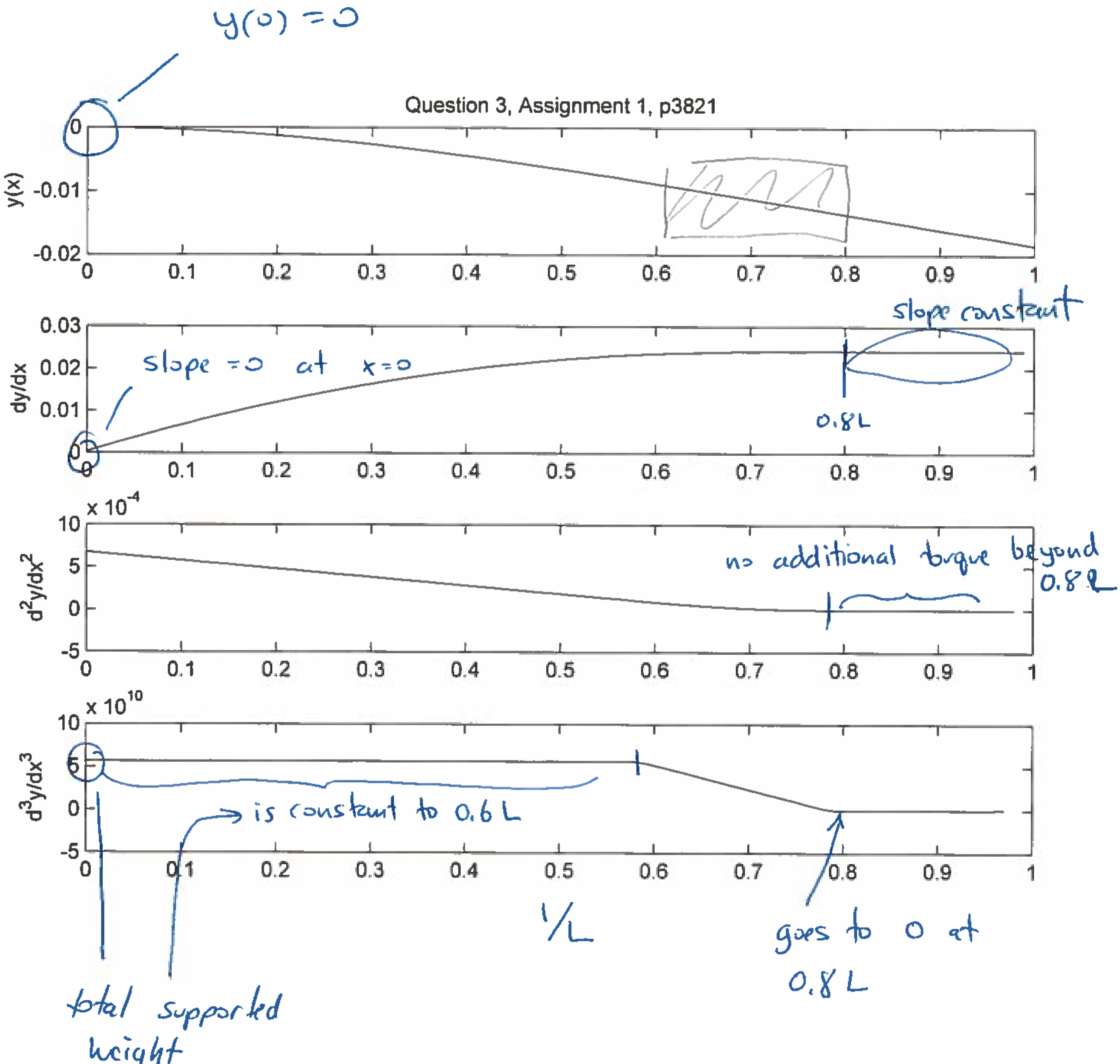
$$\cancel{y''(L)}+\frac{Mg}{EI}\left(x_0+\frac{\ell}{2}\right)=y''(0)$$

$$0$$

$$\boxed{y(x)=\frac{Mg}{EI}\cdot\left\{\frac{1}{24\ell}\left[(x-x_0)^4\,S(x-x_0)-(x-(x_0+\ell))^4\,S(x-(x_0+\ell))\right]+\left(x_0+\frac{\ell}{2}\right)\frac{x^2}{2}-\frac{x^3}{6}\right\}}$$


Question 3
0
-0.002
-0.004
-0.006
-0.008
-0.01
-0.012
-0.014
-0.016
-0.018
-0.02
Y - Displacement (1/L)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
X - Position (1/L)

```
l = .2;
L = 1.;

M = 10;
g = 9.8;
E = 1000;
I = 1;
xo = .6;

x = 0:.01:1;

yppo = M*g/E/I * (l/2 + xo);

ins = find(x<xo);
y(ins) = yppo * x(ins).^2 / 2 - M*g/E/I*x(ins).^3./6;

ins = find( (x>= xo)  & (x < (xo+l)));
y(ins) = -M*g/(l*E*I)*1/24*( ( -(x(ins)-xo).^4)) + ...
                         yppo * x(ins).^2 / 2 - M*g/E/I*x(ins).^3./6;

ins = find(x >= (xo+l));

y(ins) = -M*g/(l*E*I)*1/24*( (x(ins)-(xo+l)).^4 - ( (x(ins)-xo).^4)) + ...
                         yppo * x(ins).^2 / 2 - M*g/E/I*x(ins).^3./6;



figure(1)
clf
subplot(411)
plot(x,-y)
subplot(412)
ins = length(x);
plot(x(1:(ins-1)),diff(y)/(x(2)-x(1)))
subplot(413)
plot(x(1:(ins-2)),diff(diff(y))/ ((x(3)-x(1))/2))
subplot(414)
%plot(x(1:(ins-3)),diff(diff(diff(y))) / (( x(4) - x(2) - x(3) + x(1))/2))


%plot(x(1:(length(x)-2),-diff(diff(y)))

figure(2)
plot(x,-y)
xlabel('X - Position (1/L)')
ylabel('Y - Displacement (1/L)')
```



y(0) = 0
Question 3, Assignment 1, p3821
y(x)
dy/dx
slope = 0 at x = 0
slope constant
0.8L
x 10^-4
d^2y/dx^2
no additional torque beyond 0.8L
x 10^10
d^3y/dx^3
is constant to 0.6L
x/L
goes to 0 at 0.8L
total supported weight

④ Not asked, not completed

①

$e^{-\lambda t_{1/2}} = \frac{1}{2}$

$\lambda t_{1/2} = \ln 1/2$

$\lambda = \frac{1}{t_{1/2}} \cdot \ln 1/2$

$$\frac{dn_{Np}}{dt} = -\lambda_1 n_{Np} \qquad T_{1/2} = 2.14 \times 10^6 \, y \Rightarrow \lambda_1 = 3.24 \times 10^{-7}$$

$$\frac{dn_U}{dt} = -\lambda_2 n_U + \lambda_1 n_{Np} \qquad T_{1/2} = 1.6 \times 10^5 \, y \quad \lambda_2 = 4.33 \cdot 10^{-6}$$

$$\frac{dn_{Th}}{dt} = -\lambda_3 n_{Th} + \lambda_2 n_U \qquad T_{1/2} = 7340 \, y \quad \lambda_3 = 9.44 \cdot 10^{-5}$$

$$\frac{dn_{Bi}}{dt} = \lambda_3 n_{Th}$$

$\longrightarrow$ ~~only 98% of $^{213}Bi \rightarrow {}^{213}Po$~~ !

Converting to Laplace space:

$$s N_1(s) - n_1(0) = -\lambda_1 N_1(s)$$

$$s N_2(s) - n_2(0) = -\lambda_2 N_2(s) + \lambda_1 N_1(s)$$

$$s N_3(s) - n_3(0) = -\lambda N_3(s) + \lambda_2 N_2(s)$$

$$s N_4(s) - n_4(0) = \lambda N_3(s)$$

$N_1 = \mathcal{L}[n_{Np}]$ $\qquad n_1(0) = n_{Np}(0)$

$N_2 = \mathcal{L}[n_U]$ $\qquad$ etc. ...

$N_3 = \mathcal{L}[n_{Th}]$ $\qquad n_2(0) = 0$

$\qquad n_3(0) = 0$

$N_4 = \mathcal{L}[n_{Bi}]$ $\qquad n_4(0) = 0$

2

$$\boxed{N_1 = \frac{n_1(0)}{s+\lambda_1}}$$

$$N_2 = \frac{\lambda_1 N_1}{s+\lambda_2}$$

$$N_3 = \frac{\lambda_2 N_2}{s+\lambda_3}$$

$$N_4 = \frac{\lambda_3 N_3}{s}$$

$$\boxed{\begin{aligned} N_2 &= \frac{\lambda_1}{s+\lambda_2} \cdot \frac{n_1(0)}{s+\lambda_1} \\ N_3 &= \frac{\lambda_2}{s+\lambda_3} \cdot \frac{\lambda_1}{s+\lambda_2} \cdot \frac{n_1(0)}{s+\lambda_1} \\ N_4 &= \frac{\lambda_3}{s} \cdot \frac{\lambda_2}{s+\lambda_3} \cdot \frac{\lambda_1}{s+\lambda_2} \cdot \frac{n_1(0)}{s+\lambda_1} \end{aligned}}$$

③

Use generalised inverse:

$$n_1(t) = \frac{1}{2\pi i} \cdot \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \, \frac{n_1(0)}{s+\lambda_1} \, ds$$

simple pole at $-\lambda_1 = s$

$$\frac{1}{s+\lambda_1} \longrightarrow 0 \quad \text{as} \quad s \to \infty$$

$$= \frac{1}{2\pi i} \cdot 2\pi i \cdot \sum \text{Res's}$$

$$= \frac{1}{2\pi i} \cdot 2\pi i \cdot n_1(0) \; e^{-\lambda_1 t}$$

$$- \boxed{n_1(t) = n_1(0) \; e^{-\lambda_1 t}}$$

$$n_2(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \cdot \frac{\lambda_1}{s+\lambda_2} \cdot \frac{n_1(0)}{s+\lambda_1}$$

here we have two poles:

$s = -\lambda_2$ and $s = -\lambda_1$

$$= \frac{1}{2\pi i} \cdot 2\pi i \cdot \left[\frac{\lambda_1}{(\lambda_2 - \lambda_1)} n_1(0) e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1 - \lambda_2} n_1(0) e^{-\lambda_2 t} \right]$$

$$n_2(t) = \frac{\lambda_1 n_1(0)}{\lambda_1 - \lambda_2} \left[e^{-\lambda_2 t} - e^{-\lambda_1 t} \right]$$

$$n_3(t) = \frac{1}{2\pi i} \cdot 2\pi i \cdot \int_{\sigma - \infty i}^{i\infty} e^{st} \cdot \frac{\lambda_2}{s+\lambda_3} \cdot \frac{\lambda_1}{s+\lambda_2} \frac{n_1(0)}{s+\lambda_1} ds$$

$$= e^{-\lambda_3 t} \frac{\lambda_1 \cdot \lambda_2 n_1(0)}{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)} + e^{-\lambda_2 t} \frac{\lambda_1 \cdot \lambda_2 \cdot n_1(0)}{(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_2)} + e^{-\lambda_1 t} \frac{\lambda_1 \cdot \lambda_2 \cdot n_1(0)}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)}$$

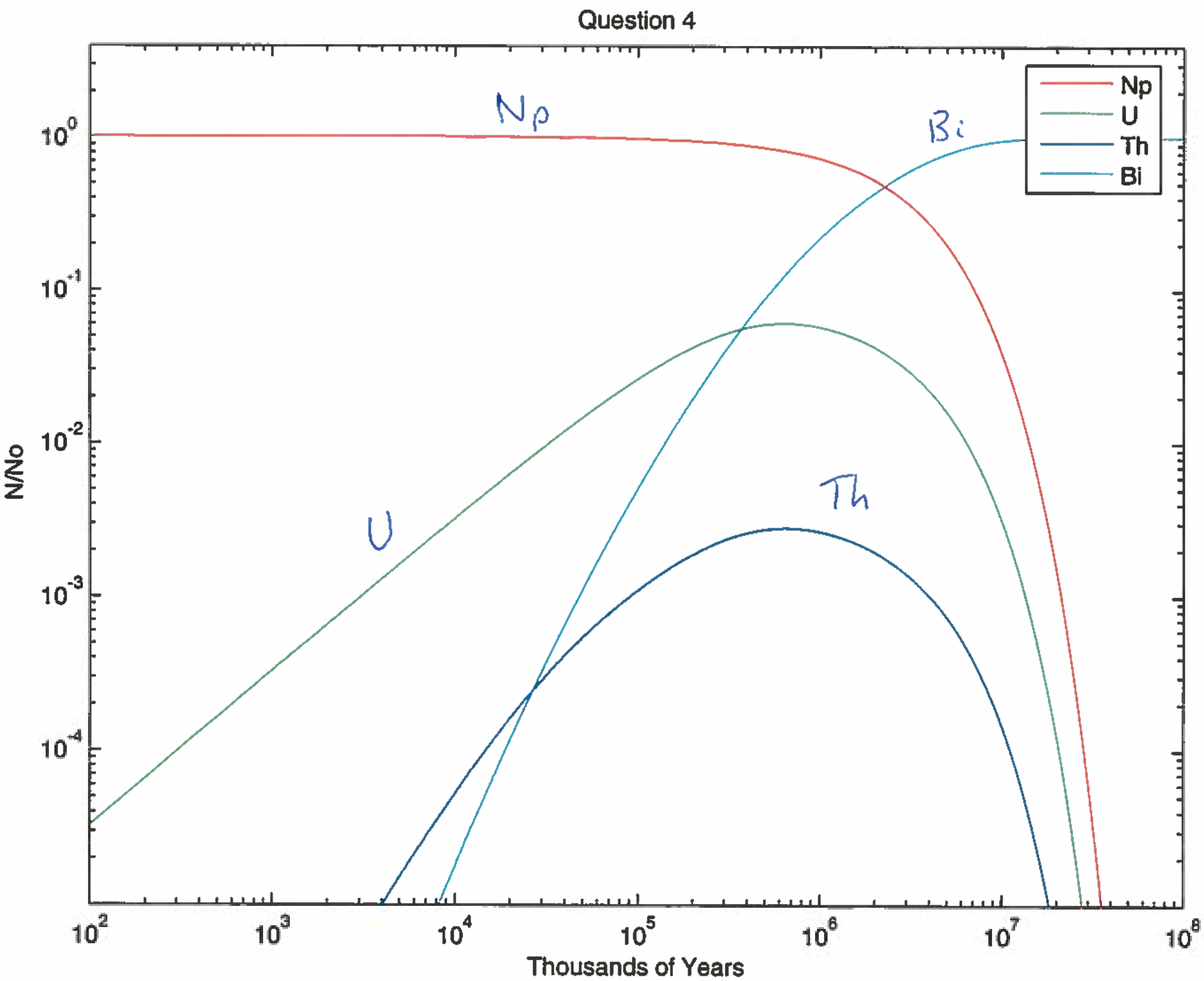
$$n_4(t) = \frac{1}{2\pi i} 2\pi i \int e^{st} \frac{\lambda_3}{s} \frac{\lambda_2}{s+\lambda_3} \frac{\lambda_1}{s+\lambda_2} \cdot \frac{n_1(0)}{s+\lambda_1} ds$$

$$= n_1(0) \cdot \cancel{\frac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3}{\lambda_3 \cdot \lambda_2 \cdot \lambda_1}} - \frac{n_1(0) \lambda_1 \lambda_2 \lambda_3 e^{-\lambda_3 t}}{\lambda_3 (\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)} + \frac{n_1(0) \lambda_1 \lambda_2 \lambda_3 e^{-\lambda_2 t}}{\lambda_2 (\lambda_3 - \lambda_2)(\lambda_1 - \lambda_2)}$$

$$- \frac{n_1(0) \lambda_1 \lambda_2 \lambda_3 e^{-\lambda_1 t}}{\lambda_1 (\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)}$$

```


nlo = 1.;

loghalf = .693147;

lambda1 = 1/2.14e6 * loghalf;
lambda2 = 1/1.6e5 * loghalf;
lambda3 = 1/7340 * loghalf;

t = 0:100:1e8;

n1 = nlo * exp(-lambda1*t);


n2 = lambda1 * nlo / (lambda1 - lambda2) * ( exp(-lambda2*t) - exp(-lambda1*t));

n31 = lambda1 * lambda2 * nlo * ( ...
        exp(-lambda3*t)./ ((lambda2 - lambda3) * (lambda1 - lambda3))+ ...
        exp(-lambda2*t)./ ((lambda3 - lambda2) * (lambda1 - lambda2)) + ...
        exp(-lambda1*t)./ ((lambda3 - lambda1) * (lambda2 - lambda1)));

n3 = lambda1 * lambda2 * nlo / ((lambda2 - lambda3) * (lambda1 - lambda3)) * exp(-lambda3 * t) + ...
     lambda1 * lambda2 * nlo / ((lambda3 - lambda2) * (lambda1 - lambda2)) * exp(-lambda2 * t) + ...
     lambda1 * lambda2 * nlo / ((lambda3 - lambda1) * (lambda2 - lambda1)) * exp(-lambda1 * t);

n4 = nlo - ...
     nlo * lambda1 * lambda2 * lambda3 / (lambda3 * (lambda2 - lambda3) * (lambda1 - lambda3)) * ...
                                                      exp(-lambda3*t) - ...
     nlo * lambda1 * lambda2 * lambda3 / (lambda2 * (lambda3 - lambda2) * (lambda1 - lambda2)) * ...
                                                      exp(-lambda2*t) - ...
     nlo * lambda1 * lambda2 * lambda3 / (lambda1 * (lambda3 - lambda1) * (lambda2 - lambda1)) * ...
                                                      exp(-lambda1*t);

figure(1)
clf
subplot(411)
plot(t/1000,n1)
ylabel('Np')
title('Question 4')
subplot(412)
plot(t/1000,n2)
ylabel('U')
subplot(413)
plot(t/1000,n3)
ylabel('Th')
```

```
subplot(414)
plot(t/1000,n4)
ylabel('Bi')
xlabel('Thousands of Years')

figure(2)
clf
loglog(t,n1,'r')
hold on
%ylabel('Np')
title('Question 4')
loglog(t,n2,'g')
%ylabel('U')
loglog(t,n3,'b')
%ylabel('Th')
loglog(t,n4,'c')
ylabel('N/No')
xlabel('Thousands of Years')
axis([100 1e8 1e-5 4])
legend('Np','U','Th','Bi')
```


Question 4
Np
U
Th
Bi
N/No
Thousands of Years
10^0
10^-1
10^-2
10^-3
10^-4
10^2
10^3
10^4
10^5
10^6
10^7
10^8