P4820 Assignment II

Due, February 15, 2019

1) [10] Show that

$$\phi_n(x) = \frac{1 - \cos nx}{n\pi x^2}$$

is a delta sequence.

2) [5] Use delta sequences to show that the limit:

$$\frac{\lim}{l \to 0} = \left[\frac{\delta(x) - \delta(x-l)}{l}\right]$$

exhibits the sifting property of $\delta'(x)$.

3) [5] A disk of radius α and mass M lies in the $x-y$ plane. Express the density in terms of delta functions in Cartesian coordinates, cylindrical coordinats, and spherical coordinates.

4) [10] A disk of charge with radius a and surface charge density $\sigma(r) = \sigma_0 r/a$ lies in the $x - y$ plane with the center at the origin. Find the volume charge density in a) cylindrical coordinates and b) spherical coordinates.

5) [10] A string of length L, with tension T and mass per unit length μ is hit simultaneously at $t = 0$ at the two points $x = L/3$ and $x = 2L/3$. The impulse delivered at each point is I. Find the equation that describes the subsequent displacement of the string. Plot the string position for times of $t = v * t/L = 1/10$ and $1/100$. Compare plots when you use the first 20 terms versus when you use the first 200 terms.

2)

$$\int_{-\infty}^{\infty}\left[\frac{1-\cos nx}{n\pi x^2}\right] f(x)\,dx = ?$$

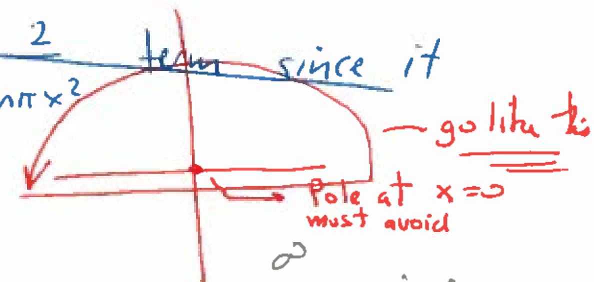
$$= \int_{-\infty}^{\infty}\frac{1}{n\pi x^2} f(x)\,dx - \int_{-\infty}^{\infty}\frac{e^{inx}+e^{-inx}}{2n\pi x^2} f(x)\,dx$$

$$\int_{-\infty}^{\infty}\frac{1}{n\pi x^2}\left(\frac{2-e^{inx}-e^{-inx}}{2}\right)f(x)\,dx$$

This we will evaluate over the complex plane,

~~no need to fuss over the~~ $\frac{2}{2n\pi x^2}$ ~~term since it~~

~~will go to zero at ∞~~



We consider

$$\underbrace{\int_{-\infty}^{\infty}\frac{1}{n\pi z^2} f(z)\,dz}_{\text{Residue}=0!} - \int_{-\infty}^{\infty}\frac{e^{inz}}{2n\pi z^2} f(z)\,dz - \int_{-\infty}^{\infty}\frac{e^{-inz}}{2n\pi z^2} f(z)\,dz$$

close on top

close on bottom

$$\frac{1}{2n\pi}\left(1 + inz - n^2z^2 + \cdots\right)$$

$$\frac{1}{z^2} + \frac{in}{z} - \cdots$$

$$\text{Res} = \frac{-in}{2n\pi} \qquad 2\pi i \cdot \frac{+in}{2n\pi} = 1$$

$$\boxed{= f(0)}$$

doesn't contain the pole at z=0

= 0

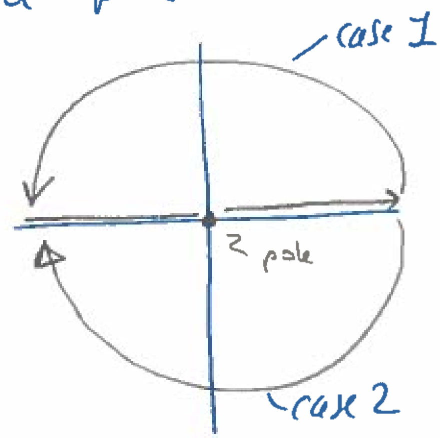
An alternative approach to the integral

$$\underbrace{\int_{-\infty}^{\infty} \frac{1}{n\pi z^2} f(z)\,dz}_{\text{Residue} = 0 \text{ no problem!}} - \int_{-\infty}^{\infty} \frac{e^{inz}}{2n\pi z^2} f(z)\,dz - \int_{-\infty}^{\infty} \frac{e^{-inz}}{2n\pi z^2} f(z)\,dz$$

(1) (2)

has pole at $x=0$! and that would be on my surface of integration if I keep the curve at $x=0$

In class we found that such a pole contributes half a residue!



$$e^{inz} = (1 + inz - n^2 z^2 + \ldots)$$

$$\frac{e^{inz}}{2n\pi z^2} = \frac{1}{2n\pi z^2} + \frac{i}{2\pi z} - \ldots$$

residue

For integral case (1), must close on top to keep integral bounded at ∞.

$$\lim_{n\to\infty} -\int_{-\infty}^{\infty} \frac{e^{inz}}{2n\pi z^2} f(z)\,dz = -\frac{1}{2}\frac{i}{2\pi}\cdot 2\pi i = \frac{1}{2} f(0)$$

case (2)

$$\int_{-\infty}^{\infty} \frac{e^{-inz}}{2n\pi z^2} f(z)\,dz = -\frac{1}{2}\frac{i}{2\pi}\cdot 2\pi i = \frac{1}{2} f(0)$$

$-$sign from $e^{-\cdots}$ but again from cw integration

Total integral

$$\int_{-\infty}^{\infty} \frac{1-\cos nx}{n\pi x^2} f(x)\, dx = \frac{f(0)}{2} + \frac{f(0)}{2} = f(0)$$

Sifting property

3\] $\lim_{\ell \to 0} \left[\frac{\delta(x) - \delta(x-\ell)}{\ell} \right]$

consider a delta sequence ϕ_n, $\lim_{n\to\infty} \phi_n \to \delta$

$$\int_{-\infty}^{\infty} \lim_{n\to\infty} \lim_{\ell\to 0} \left[\frac{\phi_n(x) - \phi_n(x-\ell)}{\ell} \right] f(x)\, dx$$

$$\lim_{n\to\infty} \lim_{\ell\to 0} \left[\frac{\int \phi_n(x) f(x)\, dx}{\ell} - \frac{\int \phi_n(x-\ell)\, f(x)}{\ell} \right]$$

$$= \lim_{\ell\to 0} \left[\frac{f(0) - f(+\ell)}{\ell} \right] = -f'(0)$$

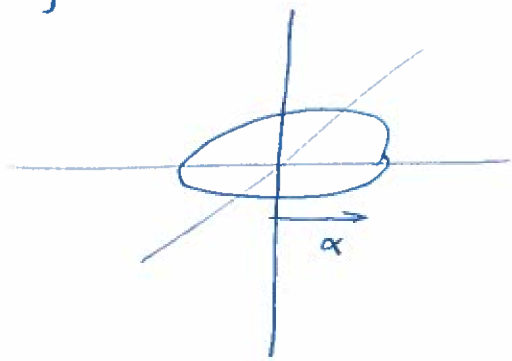
normally get $\frac{f(x+\ell) - f(x)}{\ell}$

but, $\int \delta' f(x)\, dx = -f'(0)$

$$\Rightarrow \boxed{\lim_{\ell\to 0} \frac{\delta(x) - \delta(x-\ell)}{\ell} = \delta'(x)}$$

4)

Disk mass M radius α in x,y plane
Find density!



i) Cartesian disk is at $z=0$

$$\rho(\vec{x}) = \frac{M}{\pi\alpha^2}\,\delta(z) \quad \text{for } \sqrt{x^2+y^2} < \alpha$$

and 0 otherwise:

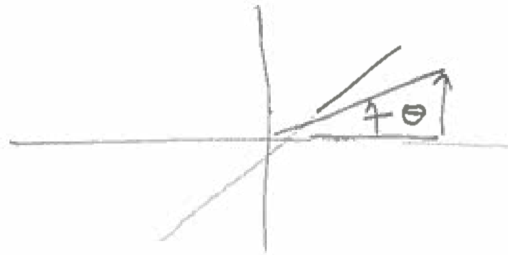
$$\boxed{\rho(\vec{x}) = \frac{M}{\pi\alpha^2}\,\delta(z)\,\Theta\left(\alpha - \sqrt{x^2+y^2}\right)}$$

step function

ii) Cylindrical coordinates
easy modification to cartesian, $\sqrt{x^2+y^2} = r$

$$\boxed{\rho(\vec{x}) = \frac{M}{\pi\alpha^2}\,\delta(z)\,\Theta(\alpha - r)}$$

iii) Spherical



$$z \Rightarrow r\cos\theta$$

$$\rho(\vec{x}) = \frac{M}{\pi a^2}\,\delta(r\cos\theta)\,\Theta(a-r)$$

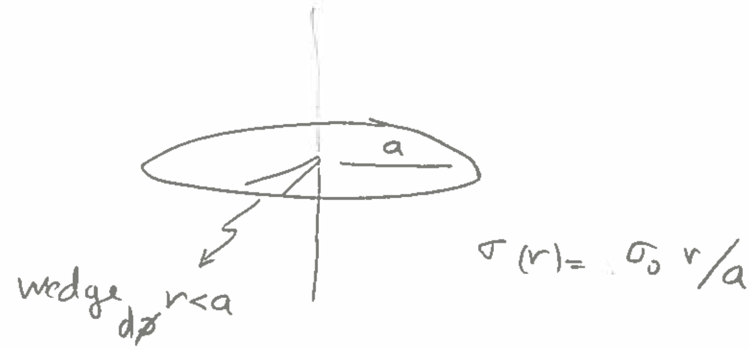
$$= \frac{M}{\pi a^2 r}\,\delta(\cos\theta)\,\Theta(a-r)$$

has a zero at $\theta = \frac{\pi}{2}$

$$= \frac{M}{\pi a^2 r}\,\frac{\delta(\theta-\frac{\pi}{2})}{\left|-\sin(\frac{\pi}{2})\right|}\,\Theta(a-r)$$

$$\boxed{\rho(\vec{x}) = \frac{M}{\pi a^2 r}\,\delta\left(\theta-\frac{\pi}{2}\right)\Theta(a-r)}$$

5)



i) In cylindrical coordinates

$$\rho(\vec{x}) = f(r,z)\,\sigma_0\,\Theta(a-r)\,\delta(z)$$

integrate over a wedge of a cylindrical shell r<a

$$dq = \sigma_0 \frac{r^2}{a}\,dr\,d\phi = \int_{-\infty}^{\infty} f(r,z)\,\sigma_0\,\Theta(a-r)\,\delta(z)\,r\,dr\,d\phi\,dz$$

$$\sigma_0 \frac{r^2}{a}\,dr\,d\phi = f(r,0)\,\sigma_0\,r\,dr\,d\phi$$

$$f(r,0) = \frac{r}{a}$$

$$\boxed{\rho(\vec{x}) = \frac{\sigma_0 r}{a}\,\Theta(a-r)\,\delta(z)}$$

ii) In polar coordinates now integrate over an orange wedge slice

absorbs r into unknown f

$$\rho(\vec{x}) = f'(r,\theta,\phi)\,\sigma_0\,\Theta(a-r)\,\delta(r\cos\theta)$$

$$= f(r,\theta)\,\sigma_0\,\Theta(a-r)\,\delta(\theta-\pi/2)$$

$$\frac{\sigma_0 r^2}{a}\,dr\,d\phi = \int_0^{\pi} f(r,\theta)\,\sigma_0\,\Theta(a-r)\,\delta(\theta-\pi/2)\,r^2\,dr\,d\phi\,\sin\theta\,d\theta$$

$$= f(r,\tfrac{\pi}{2})\,\sigma_0\;1\;r^2\,dr\,d\phi\cdot 1$$

$$f(r,\tfrac{\pi}{2}) = \frac{1}{a}$$

$$\boxed{\rho(\vec{x}) = \frac{\sigma_0}{a}\,\Theta(a-r)\,\delta(\theta-\pi/2)}$$

again, by just converting variables

$z = r\cos\theta$!

$$\rho(\vec{x}) = \frac{\sigma_0 r}{a}\,\Theta(a-r)\,\delta(z)$$

to ~~cylindric~~ spherical

$$\rho(\vec{x}) = \frac{\sigma_0 r}{a}\,\Theta(a-r)\,\delta(r\cos\theta)$$

$$= \frac{\sigma_0 \cancel{r}}{a\cancel{r}}\,\Theta(a-r)\,\delta(\cos\theta)$$

$$= \frac{\sigma_0}{a}\,\Theta(a-r)\,\frac{\delta(\theta-\pi/2)}{|-\sin\theta|}$$

$$\boxed{= \frac{\sigma_0}{a}\,\Theta(a-r)\,\delta(\theta-\pi/2)}$$

again

6) A string length L, tension T and mass density μ.

The initial condition we have at $t=0$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \frac{I}{\mu}\left[\delta\left(x-\frac{L}{3}\right)+\delta\left(x-\frac{2L}{3}\right)\right]$$

also, $y(x,0)=0$

and of course $y(0,t)=y(L,t)=0$

we seek a solution of the form:

$$y(x,t)=\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\sin\frac{n\pi v t}{L}$$

$$v=\sqrt{T/\mu}$$

gotta work from $y'(x,t)=$

$$=\left.\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\cdot\cos\frac{n\pi v t}{L}\cdot\frac{n\pi v}{L}\right|_{t=0}$$

$$\frac{I}{\mu}\left[\delta\left(x-\frac{L}{3}\right)+\delta\left(x-\frac{2L}{3}\right)\right]=\sum_{n=1}^{\infty} a_n\frac{n\pi v}{L}\cdot\sin\frac{n\pi x}{L}$$

Integrate to find a_n

$$\frac{2}{L}\frac{I}{\mu}\int_{\substack{0\\-\infty}}^{\substack{\infty\\L}} \sin\frac{n\pi x}{L}\left[\delta\left(x-\frac{L}{3}\right)+\delta\left(x-\frac{2L}{3}\right)\right]$$

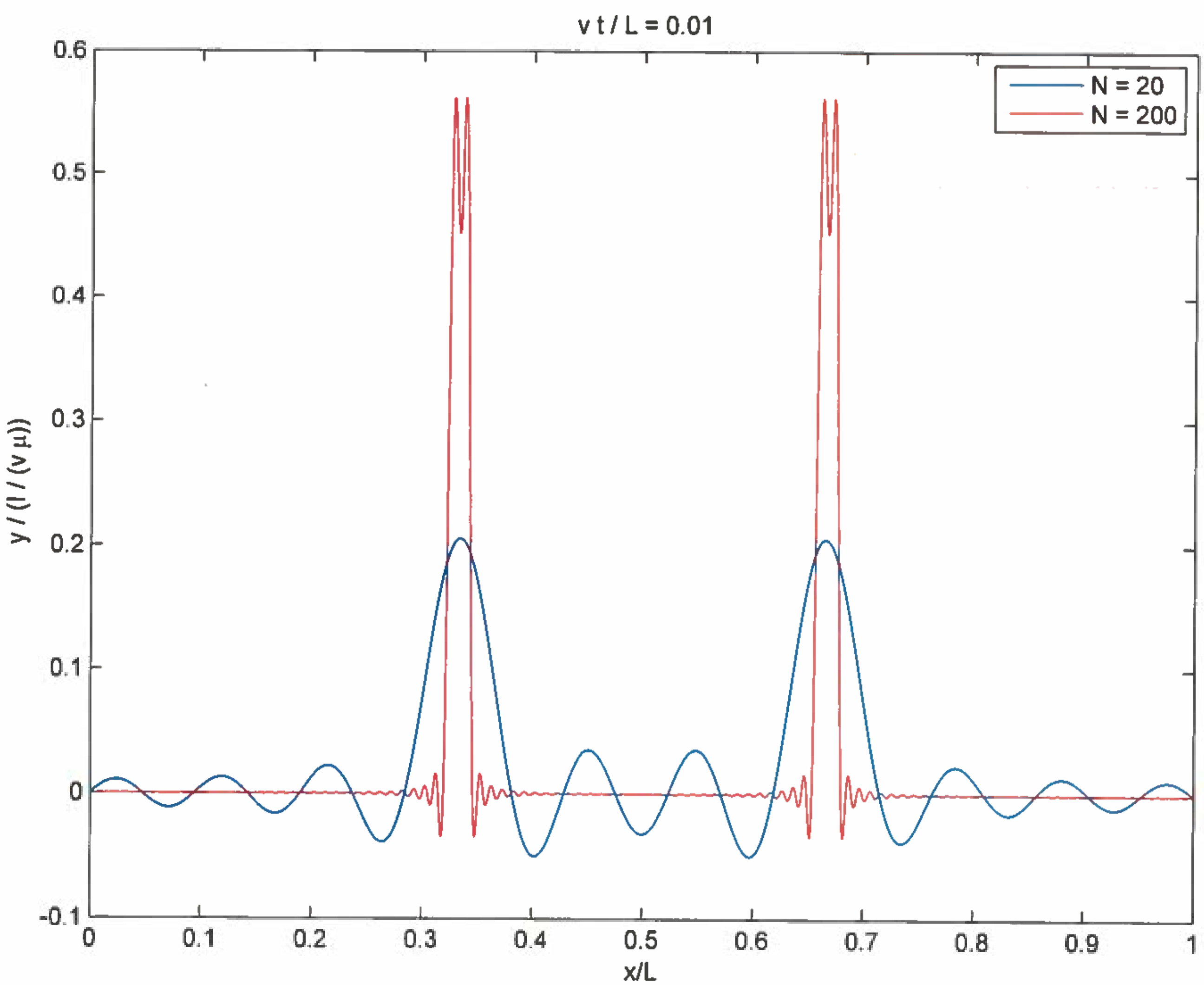
$$=\frac{2}{L}\frac{I}{\mu}\left(\sin\frac{n\pi}{3}+\sin\frac{2n\pi}{3}\right)=a_n\cdot\frac{n\pi v}{L}$$

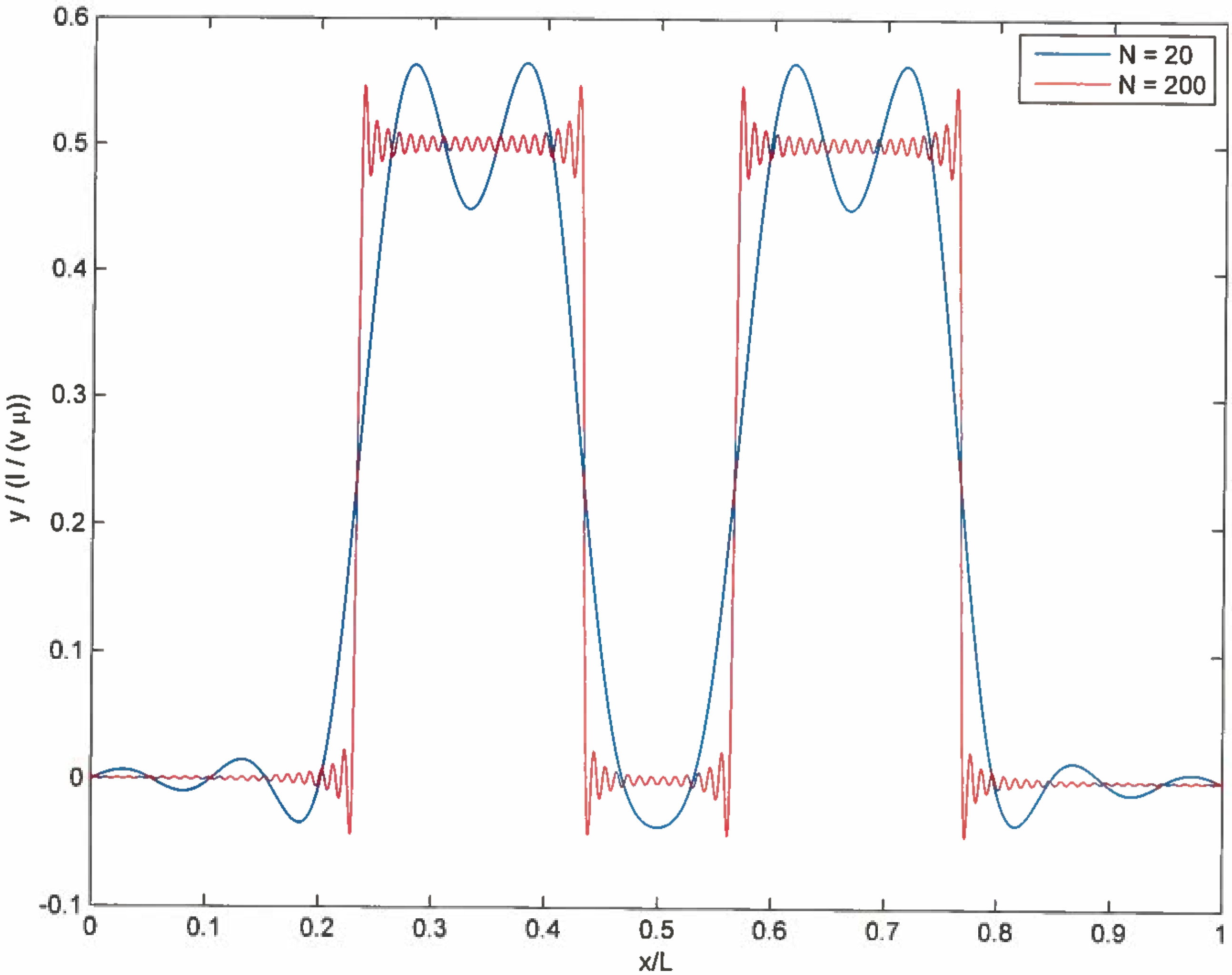
$$a_n=\frac{2}{n\pi v}\cdot\frac{I}{\mu}\left(\sin\frac{n\pi}{3}+\sin\frac{2n\pi}{3}\right)$$

$$\boxed{y(x,t)=\sum_{n=1}^{\infty}\frac{2I}{n\pi v\mu}\left(\sin\frac{n\pi}{3}+\sin\frac{2n\pi}{3}\right)\sin\frac{n\pi x}{L}\sin\frac{n\pi vt}{L}}$$

Does not really converge that fast

... $\frac{1}{n}$ term!


v t / L = 0.01
N = 20
N = 200
y / (I / (v μ))
x/L
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1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x/L