Posted notes to p 18

…need these next

Spherical Harmonics

Surrender on axial symmetry

$$\frac{d}{d\mu}\left[\overset{f}{(1-\mu^2)}\frac{dP}{d\mu}\right] - \overset{g}{\frac{m^2}{1-\mu^2}}\,P + \overset{w}{1}\,\overbrace{\ell(\ell+1)}^{\lambda}\,P = 0$$

This is SL with $f(\mu) = 1-\mu^2$ $g(\mu) = \frac{m^2}{1-\mu^2}$

$w(\mu) = 1$ $\lambda = \ell(\ell+1)$

and get $\int_{-1}^{1} P_\ell \, P_{\ell'} \, d\mu = 0$ $\ell \neq \ell'$

m fixed (ignoring a bit of what the terms came from)

But you can also construct

$$\frac{d}{d\mu}\left[\overset{f}{(1-\mu^2)}\frac{dP}{d\mu}\right] - \overset{g}{\left[-\ell(\ell+1)\right]}P + \overset{w}{\frac{-1}{1-\mu^2}}\,\overset{\lambda}{-m^2}\,P = 0$$

here, $g = -\ell(\ell+1)$ $w = \frac{-1}{1-\mu^2}$ $\lambda = m^2$

requires $\int_{-1}^{1} \frac{1}{1-\mu^2} \, P^m \, P^{m'} \, d\mu = 0$ $m \neq m'$

$\underline{\ell \text{ fixed}}$

So we want to find these P_ℓ^m

Spherical Harmonics

~~We now give up on axial symmetry~~

~~our eqn becomes~~ ~~(This is what we need to solve)~~

~~$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP}{d\mu}\right] - \frac{m^2}{1-\mu^2}P + \ell(\ell+1)P=0$~~ ✱

~~FLAG THIS EQN.~~

We had

$$\frac{d}{d\mu}(1-\mu^2)\frac{dP}{d\mu} + \ell(\ell+1)P=0$$

expand the derivatives

$$\frac{d^2P}{d\mu^2} - \mu^2\frac{d^2P}{d\mu^2} - 2\mu\frac{dP}{d\mu} + \ell(\ell+1)P=0$$

$$(1-\mu^2)\frac{d^2P}{d\mu^2} - 2\mu\frac{dP}{d\mu} + \ell(\ell+1)P=0 \qquad ②$$

and we (now) know solution is $P_\ell(\mu)$

Differentiate ② $\underline{\underline{m}}$ times and introduce

$$u(m,n,\mu) \equiv \frac{d^m}{d\mu^m}P_n(\mu)$$

and use Leibniz's rule

$$\frac{d^m}{dx^m}\left[A(x)B(x)\right] = \sum_{s=0}^{m}\binom{m}{s}\frac{d^{m-s}}{dx^{m-s}}A(x)\frac{d^s}{dx^s}B(x)$$

$$\binom{m}{s} \equiv \frac{m!}{(m-s)!\,s!}$$

get (just patiently plod through)

$$(1-\mu^2)\,u'' - 2\mu(m+1)\,u' + (\ell-m)(\ell+m+1)\,u \qquad \dagger$$

let $V(\mu) = (1-\mu^2)^{m/2}\,u = (1-\mu^2)^{m/2}\,\frac{d^m p}{d\mu^m}$

$$u = \frac{1}{(1-\mu^2)^{m/2}}\cdot V$$

$$u' = +\frac{m}{\cancel{2}}(1-\mu^2)^{-\frac{m}{2}-1}\cdot +\cancel{2}\mu V + \frac{1}{(1-\mu^2)^{m/2}}V'$$

$$u' = \frac{1}{(1-\mu^2)^{m/2}}\cdot\left[V' + \frac{m\mu V}{(1-\mu^2)}\right]$$

$$u'' = \frac{1}{(1-\mu^2)^{m/2}}\left[V'' - \frac{m\mu V}{(1-\mu^2)^2}\cdot 2\mu + \frac{mV}{(1-\mu^2)} + \frac{m\mu V'}{(1-\mu^2)}\right]$$

$$+\left[V' + \frac{m\mu V}{(1-\mu^2)}\right]\cdot\frac{-2\mu\cdot -\frac{m}{2}}{(1-\mu^2)^{m/2+1}}$$

$$u'' = (1-\mu^2)^{-m/2}\left[V'' + \frac{2m\mu V'}{1-\mu^2} + \frac{mV}{1-\mu^2} + \frac{m(m+2)\mu^2 V}{(1-\mu^2)^2}\right]$$

This merges

now plug/ replace U, U', U'' with V terms in $\dagger$

Sparing the algebra . . . ← Associated Legendre Function

$$(1-\mu^2)V'' - 2\mu V' + \left[l(l+1) - \frac{m^2}{1-\mu^2} \right] V = 0$$

which if you check back is EXACTLY of the form ✱ (p19)

AND

$$V \equiv \underbrace{P_l^m(\mu)}_{\text{associated Legendre Functions}} = \uparrow (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m} P_l(\mu)$$

sometimes $(-1)^m$ here . . . get corrected out with coefficients anyway

---

To note:

i) $P_l^m(\mu) = 0 \quad m \geq l$ (highest power of μ is μ^l)

ii) P_l is even or odd as l is even or odd

~~$(1-\mu^2)^{m/2}$ is even in μ, after m derivatives~~

~~the highest power of μ is μ^{l-m}~~

~~$P_l^m(\mu)$ is even if $\overset{l-m}{\cancel{l+m}}$ is even~~ new

~~odd odd~~

Notice that the problem ~~is symmetric in~~ has m^2 so eigenvalue ~~[illegible]~~ $-m$ gives the same ~~solution~~ eqn. and must be simply related to eqn for m.

(recall m's lineage, $\frac{1}{W}\frac{d^2W}{d\phi^2} = -m^2$)

$$\boxed{P_\ell^{-m} = (-1)^m \frac{(l-m)!}{(l+m)!} P_\ell^m}$$

$$\boxed{P_\ell^{-m} \propto P_\ell^m}$$

supposedly a ~~benefit~~ convenient?

can be shown by manipulation of Rodrigues formula w/ Leibnitz's rule

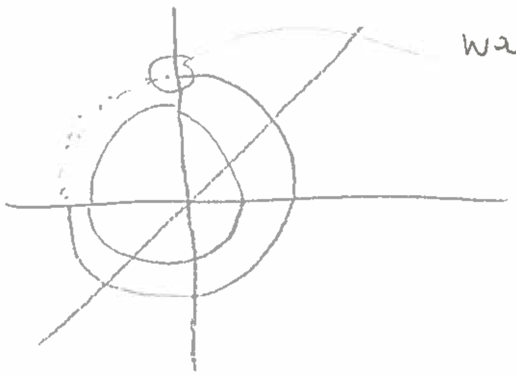
uniform in azimuth

azimuth

elevation

	$m=0$	1	2	
$\ell=0$	1	/	/	/
$\ell=1$	μ	$-\sqrt{1-\mu^2}$	/	/
$\ell=2$	$\frac{1}{2}(3\mu^2-1)$	$-3\mu(1-\mu^2)$	$3(1-\mu^2)$	/
$\ell=3$	$\frac{\mu}{2}(5\mu^2-3)$	$-\frac{3}{3}(5\mu^2-1)(1-\mu^2)^{1/2}$	$15\mu(1-\mu^2)$	

~~[illegible]~~ $\ell=0 \Rightarrow$ uniform in ~~[illegible]~~ elevation requires uniform in azimuth!



would have discontinuities

– Need to know orthogonality   what about $\ell \rightleftharpoons \ell'$?

$$I_{\ell m} = \int_{-1}^{1} P_\ell^m(\mu)\, P_{\ell'}^m(\mu)\, d\mu = \int_{-1}^{1} (1-\mu^2)^m \frac{d^m P_\ell}{d\mu^m} \frac{d^m P_{\ell'}}{d\mu^m}\, d\mu$$

$\hookrightarrow$ uses $P_\ell^m(\mu) = (-1)^m (1-\mu^2)^{m/2} \dfrac{d^m}{d\mu^m} P_\ell(\mu)$

$$I_{\ell m} = \frac{(\ell+m)!}{(\ell-m)!} \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

also

$$\boxed{\int_{-1}^{1} \frac{(P_\ell^m)^2}{1-\mu^2}\, d\mu = \frac{1}{m} \frac{(\ell+m)!}{(\ell-m)!} \quad \cancel{\delta_{mm'}}}$$

Gives

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[a_{\ell m} r^\ell + \frac{b_{\ell m}}{r^{\ell+1}} \right] P_\ell^m(\mu)\, e^{im\phi}$$

↑ associated Legendre fns.

$\hookrightarrow$ general solution to $\nabla^2\phi$ in sperical coordinates

... notice the normalisation is a bit funcky...

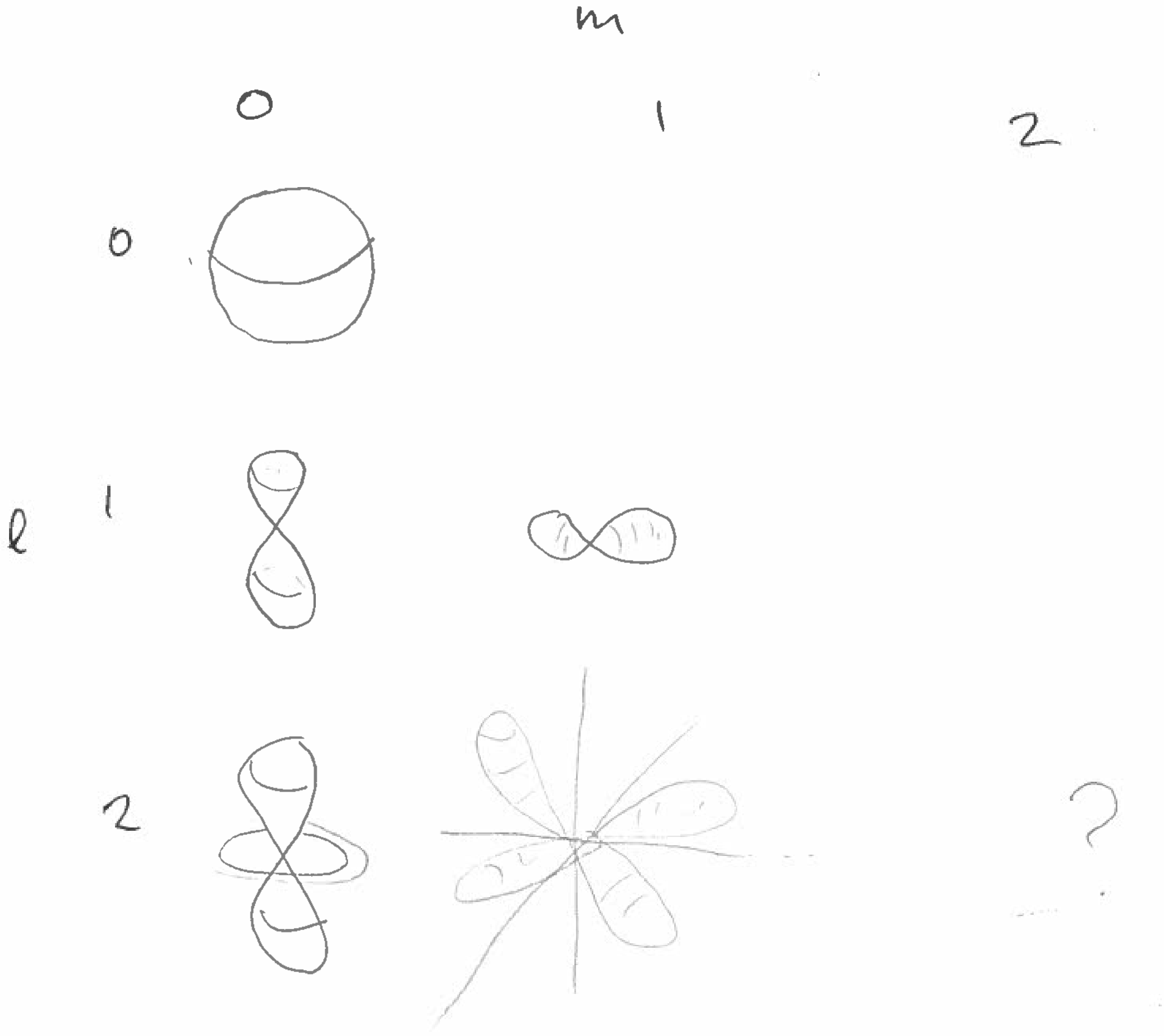
wouldn't it be nice to have a straight normal...
ortho-normal

Try

$$\underbrace{\int_{-1}^{1}\int_{0}^{2\pi}}_{\text{spherical surface}} Y_{\ell m}(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi)\, d\phi\, d\mu = \delta_{\ell\ell'} \delta_{mm'}$$

$$\boxed{\begin{gathered} Y_{\ell m}(\theta,\phi) \equiv \left[\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\mu)\, e^{im\phi} \\ \iint Y_{\ell m}\, Y^*_{\ell' m'}\, d\phi\, d\mu = \delta_{\ell\ell'}\,\delta_{mm'} \end{gathered}}$$

gives spherical structure . . .



bottom p107

General solution for $\nabla^2\phi$ in spherical domain

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm} r^{l} + \frac{B_{lm}}{r^{l+1}}\right] Y_{lm}(\theta,\phi)$$

Problems are solved again in same way assuming solution as summation and solve for coefficients using orthogonality

$$\int_{-1}^{1}\int_{0}^{2\pi} Y_{lm} Y^{*}_{l'm'}\, d\phi\, d\mu = \delta_{ll'}\,\delta_{mm'}$$

(both l & m must match)

Solve by knowing value on surface

$$\iint \sum\sum \left[A_{lm} r^{l} + B_{lm}\frac{1}{r^{l+1}}\right] Y_{lm} Y^{*}_{l'm'}\, d\theta\, d\phi$$

$$= \int\int \left[\text{value on surface}\right] Y^{*}_{l'm'}(\theta,\phi)\, d\theta\, d\phi$$

Bessel Functions

Problems with cylindrical symmetry

Need solutions to Laplaces equation in cylindrical coordinates!

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial\phi}{\partial\rho}\right]+\frac{1}{\rho^2}\frac{\partial\ \Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

Let $\Phi = R(\rho)\, W(\phi)\, Z(z)$

and divide by RWZ

$$\underbrace{\frac{1}{R\rho}\frac{d}{d\rho}\left[\rho\frac{dR}{d\rho}\right]+\frac{1}{W\rho^2}\frac{d^2W}{d\phi^2}}_{\rho\ \&\ \phi}+\underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{Z\text{ only}}=0$$

Assume separation constant $k>0$
(k<0 will give modified Bessel fn)

$$\frac{d^2Z}{dz^2}=k^2Z \longrightarrow \boxed{Z=e^{\pm kz}}$$

Now ρ, ϕ problem

$$\frac{1}{R\rho}\frac{d}{d\rho}\left[\rho\frac{dR}{d\rho}\right] + \frac{1}{W\rho^2}\frac{d^2W}{d\phi^2} + k^2 = 0$$

$$\underbrace{\frac{\rho}{R}\frac{d}{d\rho}\left[\rho\frac{dR}{d\rho}\right]} + \underbrace{\frac{1}{W}\frac{d^2W}{d\phi^2}}_{\phi\ \text{only}} + \underbrace{\rho^2k^2} = 0$$

ρ only

Choose separation constant m^2 as < 0 to force periodic solutions ... remember geometry <u>cylinder</u> ... when you've gone 360° you have to end up at the same place again.

$$\frac{d^2W}{d\phi^2} = -m^2W \rightarrow \boxed{W = e^{\pm im\phi}}$$

m controls form

leaving the R part ...

for circular shapes $m=0$!

$(\cancel{k}^2 + k^2)$ For Helmholtz (wave)

$$\rho \frac{d}{d\rho}\left[\rho \frac{dR}{d\rho}\right] + k^2 \rho^2 R - m^2 R = 0$$

$$\frac{d}{d\rho}\left[\rho \frac{dR}{d\rho}\right] + k^2 \rho R - \frac{m^2}{\rho} R = 0$$ Z1

SL $f(\rho) = \rho \quad g(\rho) = \frac{m^2}{\rho} \quad \lambda = k^2 \quad W(\rho) = \rho$

Change variables $\Longrightarrow$ $x = k \cdot \rho$ [dimensionless]

$$\rho = \frac{x}{k} \qquad \frac{d}{d\rho} = k \frac{d}{dx}$$ $dx = k\,d\rho$

$$k \frac{d}{dx}\left[\frac{x}{\cancel{k}} \cdot \cancel{k} \frac{dR}{dx}\right] + \frac{k^{\cancel{2}}\, x}{\cancel{k}} \cdot R - \frac{m^2 k}{x} R = 0$$

k falls out!

$$\boxed{\frac{d}{dx}\left[x \frac{dR}{dx}\right] + x R - \frac{m^2}{x} R = 0}$$

$x = 0$ a problem

here

look for solutions of form: (method of Froebenius)

$$R = x^p \sum_{n=0}^{\infty} a_n x^n$$

$$(x - x_0)^p \sum a_n (x - x_0)^n$$

p .. not integer .. very general

Substitute into eqn . . . .

and require all powers $= 0$ independently

For $a_0 \neq 0 \Rightarrow \boxed{p = \pm m}$ ← Flag the $+m$ . . . . . .

recursion relation:

$$\boxed{a_t = \frac{-a_{t-2}}{t(t \pm 2m)}}$$

First take $p = m$ start with a_0 and get even k's

$$a_{2n} = \frac{-1}{2n(2n+2m)} a_{2n-2}$$

$$= \frac{-1}{2n(2n+2m)} \cdot \frac{-1}{(2n-2)(2n-2+2m)} a_{2n-4} \cdots$$

2^n 2^n

$$= a_0 \frac{(-1)^n}{2^{ⓝ} n!} \cdot \frac{1}{\underbrace{2^{ⓝ}(n+m)(n+m-1)\cdots m+1}_{(a)}}$$

(in general)

lets choose (a convention) $a_0 = \frac{1}{2^m \Gamma(m+1)}$

Trick is to notice (a) goes to $(m+1)$ but a_0 carries on $m!$ gives $(n+m)!$

$$a_{2n} = \frac{(-1)^n}{n!\,\Gamma(n+m+1)\,2^{2n+m}}$$

Giving

$$\boxed{J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(\underset{p}{n+m+1})} \left(\frac{x}{2}\right)^{\overset{p}{m+2n}}}$$

k·p

← Eqn for ρ dependence!

Bessel Function

kρ

$$W = e^{\pm im\phi} \quad \text{angle}$$

$$Z = e^{\pm kz} \quad z$$

eigen value in z!

J_ν is okay too ν non integer!

AND remember we chose $p = \pm m$

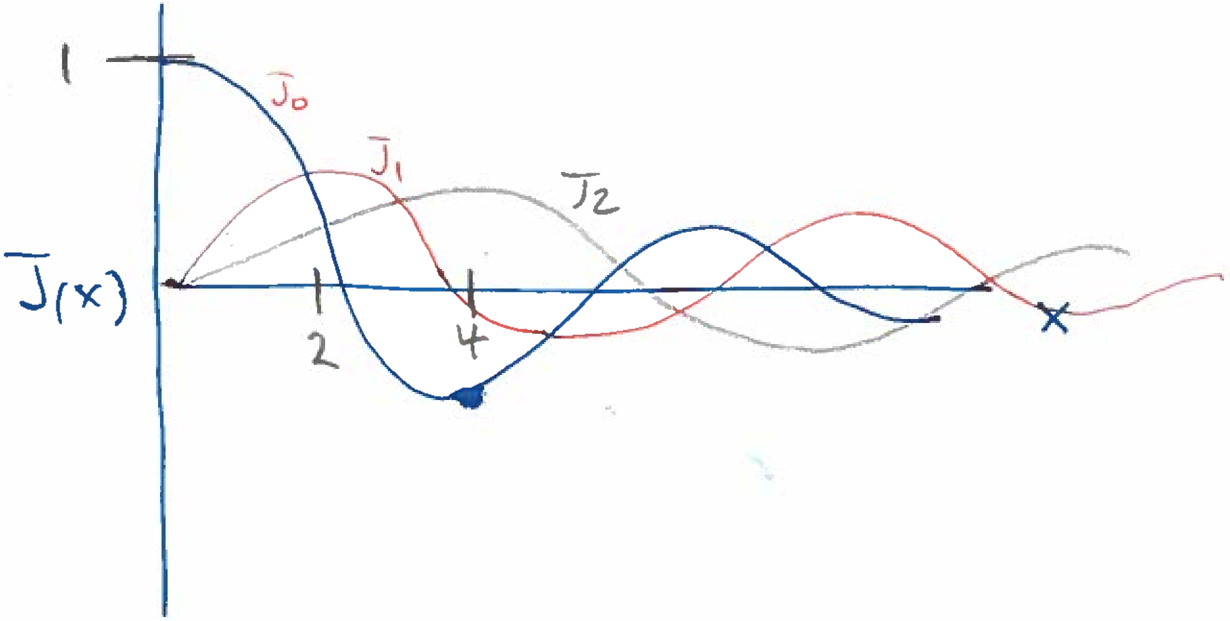
... if got take $-m$

$$\boxed{N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}}$$

a second linearly independent solution

→ $N \to \infty$ as $\rho \to 0$

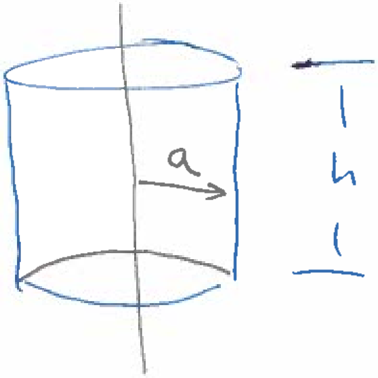
What do they look like ..



- They aren't periodic (like sin/cos)
  BUT they have repeated 0's → infinite

Example cylindrical resonant cavity . . .

EM waves will bounce up and down ... what are the resonant frequencies?



For large $\underline{\underline{x}}$

$$J_m(x) \simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

$$N_m(x) \simeq \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

attenuated sin/cos

If we had taken $\frac{\partial^2 Z}{\partial z^2} = -k^2 Z$

$\boxed{\text{get solutions periodic in } Z}$

Alters our solution in R

Modified Bessel Eqn

$$\boxed{\frac{d}{dx}\left[x \frac{dR}{dx}\right] \underset{\uparrow}{-} x R - \frac{m^2}{x} R = 0}$$

had + here before

Solutions are $J_m(ik\rho)$ and $N_m(ik\rho)$

$$I_m \equiv \frac{1}{i^m} J_m(ik\rho)$$

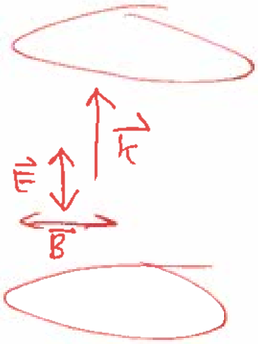
$$K_m = \frac{\pi}{2} i^{m+1}\left[J_m(ik\rho) + i N_m(ik\rho)\right]$$

Sadly these the form for Ass 4, Q2

$$\nabla^2 E_z = -\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

Assumes E_z component only

($B_z = 0$ ....



also possible to have

→ we seek the transverse magnetic ~~ss~~ wave

standing wave solutions of form

$$E_z = E(\rho,\theta,z)\cdot f(t)$$

$f(t)$ goes as $\underset{\overset{\uparrow}{\equiv}}{\sin\omega t}$, $\cos\omega t$

Separation of variables will give

$$\nabla_c^2 E_z + k^2 E_z = 0 \qquad k = \frac{\omega}{c}$$

with conducting boundaries

$E_x = E_y = 0$ at $z=0$, $z=h$ $\longleftarrow$ ☆

$E_z = 0$ on $\rho = a$

For E_z :

$$E_z(\rho, \theta, z) = P_\ell^m(\rho)\, \Phi_m(\phi)\; Z_\ell(z)$$

will have form $\Phi_m = e^{\pm im\phi}$

or $\sin m\phi$

We've done this before:

$$\frac{d^2 Z_\ell}{dz^2} = -k^2 Z_\ell$$

harmonic oscillator

And

Bessel

$$\rho \frac{d}{d\rho}\left[\rho \frac{dP_\ell^m}{d\rho}\right] + \underbrace{(k^2 - K^2)}_{\text{we had } k^2 \text{ here before alone}} \rho\, P_\ell^m = 0$$

check indices * ✓

$$\boxed{k^2 - K^2 = n^2}$$

$$E_z = J_m(\underbrace{n\rho}_{\text{before } \boxed{x = k\rho}})\, e^{\pm im\phi}\left[A \sin \ell z + B \cos \ell z\right]$$

J_m because $J_m(0) =$ [illegible] finite!

m is integer to ~~get~~ keep cos, sin form

On the boundary at $\rho = a$

$$J_m(na) = 0$$

So here we have a problem ... the zeros of J_m are not periodic!

$\alpha_{mj} \equiv$ j'th zero of J_m

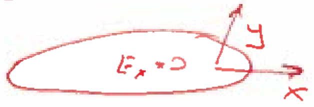
So then

$$na = \alpha_{mj} \rightarrow n^2 = \bar{k}^2 - k^2$$

$$\rightarrow \bar{k}^2 - k^2 = \left(\frac{\alpha_{mj}}{a}\right)^2$$

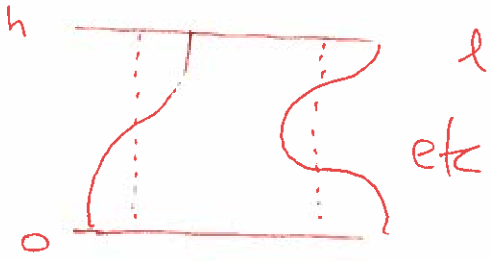
For Z term ... we know $\nabla \cdot \vec{E} = 0$

but E_x and $E_y = 0$ on $z=0$, $z=h$



$$\underbrace{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}}_{\boxed{=0}} + \frac{\partial E_z}{\partial z} \Rightarrow \boxed{\frac{\partial E_z}{\partial z} = 0}$$

Requires Z of form $B \cos lz$



$l = \frac{p\pi}{h}$ $p = 0, 1, 2 \cdots$

$$\bar{k}^2 = \frac{\omega^2}{c^2} = \left(\frac{\alpha_{mj}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2$$

So resonant frequencies are:

$$c\sqrt{\frac{\alpha_{mj}^2}{a^2} + \frac{p^2\pi^2}{h^2}} = \omega$$

bessel

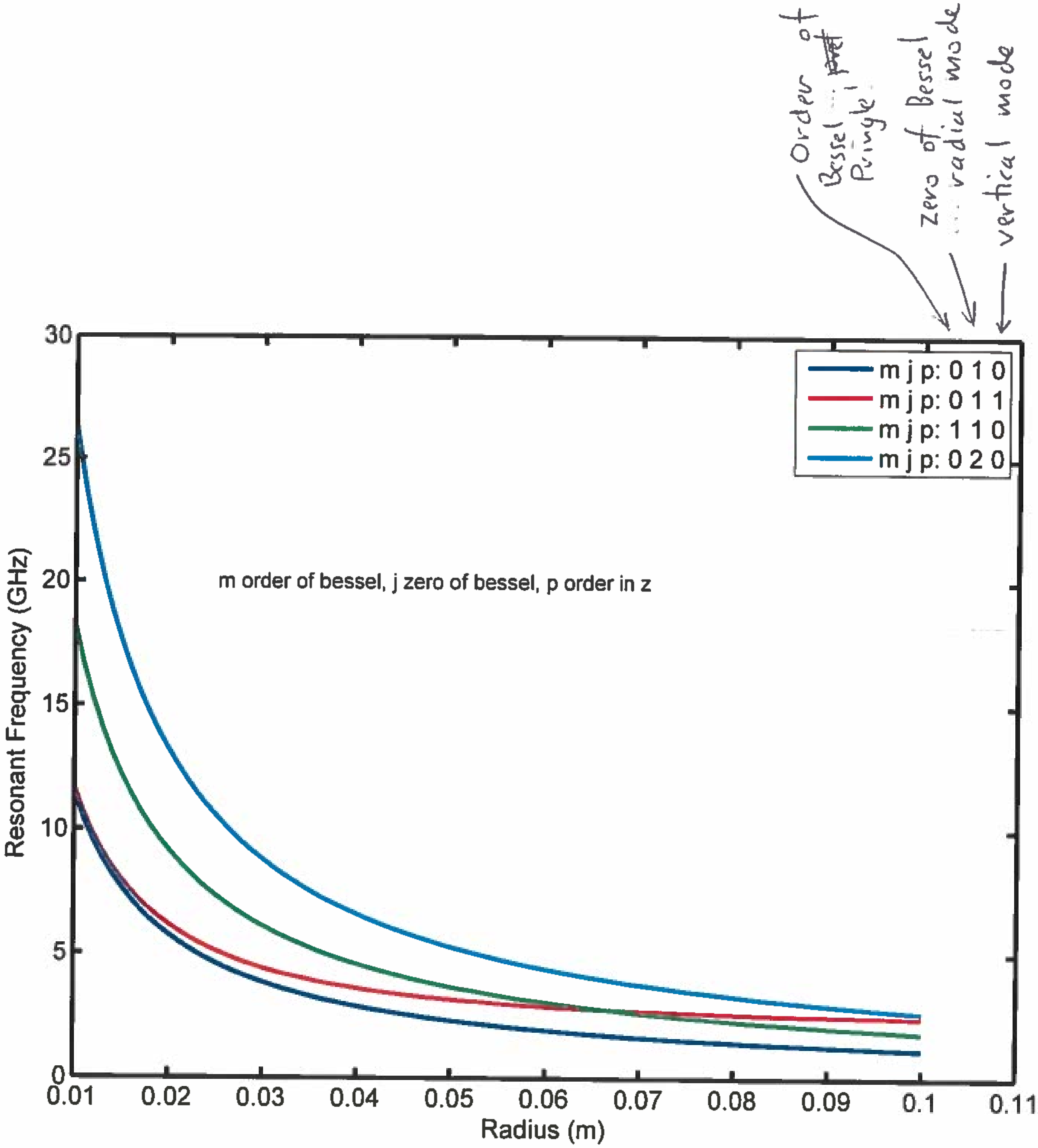
radial 'mode'

p vertical mode

as $a \rightarrow \infty$

$$\boxed{c\frac{p\pi}{h} = \omega}$$

and notice $m \neq 0 \Rightarrow$ Pringle twists through $e^{im\phi}$!


Order of Bessel Pringle!
Zero of Bessel radial mode
vertical mode
m j p: 0 1 0
m j p: 0 1 1
m j p: 1 1 0
m j p: 0 2 0
m order of bessel, j zero of bessel, p order in z
Resonant Frequency (GHz)
0
5
10
15
20
25
30
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
0.11
Radius (m)

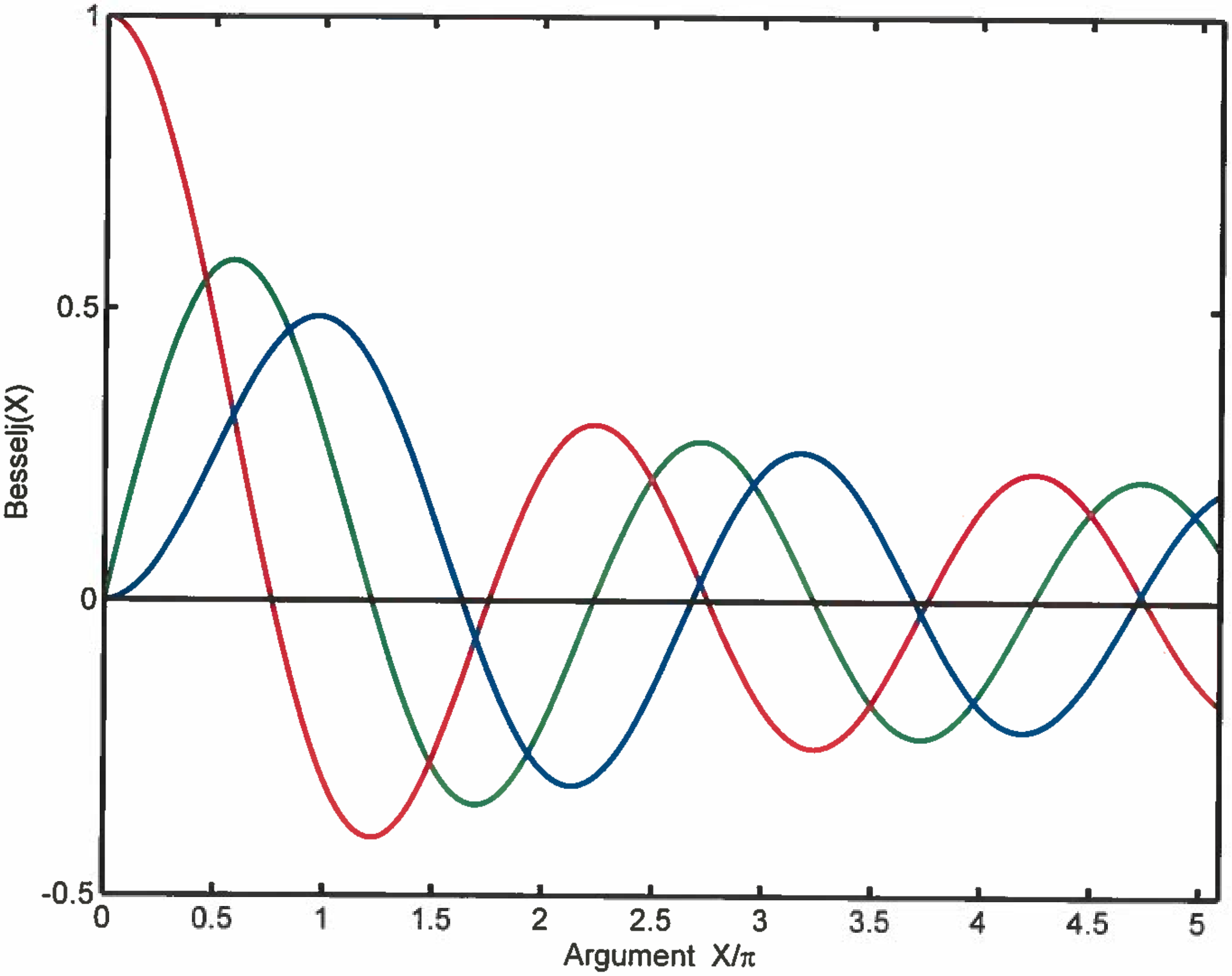
7 cm height

```
function [BZ] = bessel_zeros(m,n);

% m is order of bessel
% n is number of zeros

for ss = 1:n
  clos(ss) = (m/2 + 3/4 + (ss - 1)) * pi;
  BZ(ss) = fzero(@(z) besselj(m, z), clos(ss));
end

ii = 7;
```

$$\cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) = 0$$

$$x - \frac{m\pi}{2} - \frac{\pi}{4} = \frac{n}{\pi}$$

✓ $$x = \left(\frac{m}{2} + \frac{1}{4} + \frac{n}{2}\right)\pi$$


1
0.5
0
-0.5
Besselj(X)
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Argument X/π

```
p = 0;
j = 1;  % 1
m = 1;

il = 0;
for a = astart:.001:aend;
    il = il + 1;
    besselzeros = bessel_zeros(m,j);
    alphamj = besselzeros(j);

    f(il) = 3e8 / 2 / pi * sqrt(alphamj^2/a^2 +  p^2*pi^2/h/h) ;
end

a = astart:.001:aend;
plot(a,f/1e9,'g','LineWidth',2); hold on

p = 0;
j = 2;
m = 0;

il = 0;
for a = astart:.001:aend;
    il = il + 1;
    besselzeros = bessel_zeros(m,j);
    alphamj = besselzeros(j);

    f(il) = 3e8 / 2 / pi * sqrt(alphamj^2/a^2 +  p^2*pi^2/h/h) ;
end

a = astart:.001:aend;
plot(a,f/1e9,'c','LineWidth',2); hold on



xlabel('Radius (m)')
ylabel('Resonant Frequency (GHz)')
legend('m j p: 0 1 0','m j p: 0 1 1','m j p: 1 1 0','m j p: 0 2 0')
text(.025,20,'m order of bessel, j zero of bessel, p order in z')
```