Sturm - Liouville Theory

Linear ODE of form

$$\frac{d}{dx}\left[f(x)\frac{dy}{dx}\right] - g(x)\cdot y + \lambda \underset{\substack{\uparrow\\ \text{weight function}}}{w(x)}\, y = 0$$

$w(x) \geq 0$

some finite interval $a \leq x \leq b$

boundary values

$$\alpha_1 y + \beta_1 \frac{dy}{dx} = 0 \qquad x = a$$

$$\alpha_2 y + \beta_2 \frac{dy}{dx} = 0 \qquad x = b$$

Solutions are formed for a set of allowed values $\underline{\lambda}$

$\alpha = 0 \Rightarrow$ b.c. is $\frac{dy}{dx} = 0$ <u>Neuman</u> condition

$\beta = 0 \Rightarrow$ b.c. is $y = 0$ <u>Dirichlet</u> condition

The string suspended at both ends is an SL problem

$$v^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \qquad y = X(x)\, T(t)$$

separates X, T

$$\frac{d^2 X}{dx^2} + k^2 X = 0 \qquad \frac{d^2 T}{dt^2} + k^2 v^2 T = 0$$

$$X(0) = X(L) = 0 \quad \leftarrow \text{Dirichlet}$$

So for X we have SL with $f(x) \equiv 1$ $g(x) = 0$
$w(x) = 1$

The solution is a set of sine functions

$$k = \boxed{\frac{n\pi}{L}} \qquad \sin kx = \boxed{\sin \frac{n\pi x}{L}}$$

eigenvalues eigen functions

The solutions have the fascinating capability that they exhibit orthogonality

$$\int_0^L \sin\frac{n\pi x}{L} \cdot \sin\frac{m\pi x}{L}\, dx = \frac{L}{2}\,\delta_{mn}$$

↳ This is a characteristic of SL problems!

again say

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -m^2$$

note - sign moved

$$\Rightarrow \quad \frac{1}{Z}\frac{d^2 Z}{dz^2} - \ell^2 + k^2 = +m^2$$

or

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = \ell^2 - k^2 + m^2 = -n^2$$

This gives 3 ODE's!

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\ell^2$$

$$\frac{1}{Y}\frac{d^2 Y}{dy^2} = -m^2$$

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -n^2$$

!

with ~~$\ell^2 - k^2 + m^2 = -n^2$~~ $\Rightarrow \underline{\underline{k^2 = \ell^2 + m^2 + n^2}}$

And

$$\psi_{\ell m n} = X_\ell \, Y_m \, Z_n$$

general solution

$$\psi = \sum_{\ell m n} a_{\ell m n} \psi_{\ell m n}$$

forced to meet b.c.s!

Revisit Separation of Variables

(for example)

Helmholtz eqn

$$\nabla^2 \psi + k^2 \psi = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0$$

Just go out on a limb and say

$$\psi = X(x)\, Y(y)\, Z(z)$$

$$\nabla^2 \psi = \frac{d^2 X}{dx^2} Y Z + \frac{d^2 Y}{dy^2} X Z + \frac{d^2 Z}{dz^2} X Y + k^2 XYZ = 0$$

divide by XYZ

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} + k^2 = 0$$

$$\underbrace{\frac{1}{X}\frac{d^2 X}{dx^2}}_{\text{function of } X} = \underbrace{-k^2 - \frac{d^2 Y}{dy^2}\frac{1}{Y} - \frac{d^2 Z}{dz^2}\frac{1}{Z}}_{\text{function of } y, z}$$

$\Longrightarrow$ how can this be?

hmmm . . .

let $$\boxed{\frac{1}{X}\frac{d^2X}{dx^2} = -\ell^2}$$ ← separation constant . . . . only way it can work!

~~Seq~~ requires

$$-k^2 - \frac{d^2Y}{dy^2}\frac{1}{Y} - \frac{d^2Z}{dz^2}\frac{1}{Z} = -\ell^2$$

$$\underbrace{\frac{d^2Y}{dy^2}\frac{1}{Y}}_{\text{only depends on } y} = \underbrace{-k^2 + \ell^2 - \frac{1}{Z}\frac{d^2Z}{dz^2}}_{\text{only depends on } z\,!}$$

repeat $$\boxed{\frac{1}{Y}\frac{d^2Y}{dy^2} = -m^2}$$

leaving $$\frac{1}{Z}\frac{d^2Z}{dz^2} = m^2 + \ell^2 - k^2 \equiv -n^2$$

We have 3 ODE's

$$\frac{1}{X}\frac{d^2X}{dx^2} = -l^2$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -m^2$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -n^2$$

And $k^2 = l^2 + m^2 + n^2$

Then our solution is

$$\psi_{lmn} = X_l Y_m Z_n$$

and the general solution

$$\psi = \sum_{lmn} \underset{\uparrow \text{ constant chosen to preserve b.c.}^s}{a_{lmn}} \psi_{lmn}$$

Orthogonality

Consider 2 sets of SL solutions y_n, y_m with corresponding eigenvalues λ_n, λ_m valid on interval a, b

$$\frac{d}{dx}\left[f(x)\frac{dy_n}{dx}\right] - g(x)\,y_n + \lambda_n w(x)\,y_n = 0 \quad ①$$

and
$$\frac{d}{dx}\left[f(x)\frac{dy_m}{dx}\right] - g(x)\,y_m + \lambda_m w(x)\,y_m = 0 \quad ②$$

What we need to show is

$$\int_a^b w(x)\,y_n\,y_m = 0$$

clearly we have to get y_n & y_m together

~~Subtract the two eqns 2 − 1~~

mpy 2 by y_n and 1 by y_m

$$y_m \frac{d}{dx}\left[f(x)\frac{dy_n}{dx}\right] - g(x)\,y_n y_m + \lambda_n w(x)\,y_n y_m = 0$$

$$y_n \frac{d}{dx}\left[f(x)\frac{dy_m}{dx}\right] - g(x)\,y_m y_n + \lambda_m w(x)\,y_n y_m = 0$$

And subtract two eqns ...

$$y_m \frac{d}{dx}\left[f(x)\frac{dy_n}{dx}\right] - y_n \frac{d}{dx}\left[f(x)\frac{dy_m}{dx}\right] + 0 + (\lambda_n - \lambda_m)\, w(x)\, y_n y_m = 0$$

now integrate $\int_a^b$

$$\int_a^b \left[y_m \frac{d}{dx}\left(f(x)\frac{dy_n}{dx}\right) - \underbrace{y_n \frac{d}{dx}\left(f(x)\frac{dy_m}{dx}\right)}_{\oplus}\right] dx = (\lambda_m - \lambda_n)\int w(x)^{+}\, y_n y_m \, dx$$

↙ integrate by parts

$$\underbrace{y_m\, f(x)\, \frac{dy_n}{dx}\Big|_a^b}\; - \underbrace{\int_a^b f(x)\, \frac{dy_n}{dx}\, \frac{dy_m}{dx}\, dx}$$

either Dirichlet
or Neuman b.c.'s
kill this !

this term is generated
in the same way but
of course − sign from ⊕

stand by

$$\rightarrow\; y_m(b)\, f(b)\, y_n'(b) - y_m(a)\, f(a)\, y_n'(a) \qquad *$$

suppose $\alpha_1 y + \beta_1 y'\big|_a = 0$

$$\alpha_1 y = -\beta_1 y'$$

$$y' = -\frac{\alpha_1}{\beta_1}\, y\Big|_a \qquad y' = -\frac{\alpha_2}{\beta_2}\, y\Big|_b$$

populate *

$$y_m(b)\, f(b) \cdot \frac{-\alpha_2}{\beta_2}\, y_n(b) - y_m(a)\, f(b) \cdot \frac{-\alpha_1}{\beta_1}\, y_n(a)$$

these two terms are symmetrical in y_n y_m so the ~~So~~ second integral ⊕ will cancel these terms!

and so

$$0 = (\lambda_m - \lambda_n) \int_a^b w(x)\, y_m y_n \, dx$$

So either $\lambda_m = \lambda_n$ OR $\int_a^b$ is 0!

The integral can disappear for other reasons

i) If $f(x)\Big|_{a,b} = 0$ then (for finite y) you'd get the same answer ... this is Legendre's equation

ii) if yy' is periodic on $b-a$ also

Completeness

If you find a set of eigen functions, they form a complete orthogonal set on $[a,b]$ which means for any function $f(x)$ ← nice

$$\boxed{f(x) = \sum_{n=0}^{\infty} a_n y_n(x)}$$

and this property is a consequence of orthogonality

$$a_n = \frac{\int_a^b f(x)\, y_n(x)\, w(x)\, dx}{\int_a^b y_n^2(x)\, w(x)\, dx}$$

with property

$$\lim_{N\to\infty} \int_a^b \left[f(x) - \sum_{n=0}^{N} a_n y_n(x) \right]^2 w(x)\, dx = 0$$

converges in the mean

Completeness (always works for SL solutions)

↳ of ability to create series solutions of SL eigen functions.

$$f(x) = \sum_{n=0}^{\infty} a_n y_n = \sum_{n=0}^{\infty} \underbrace{\left[\frac{\int_a^b f(x')\, y_n(x')\, w(x')\, dx'}{I_n} \right]}_{\text{not a f of x!}} y_n(x)$$

with $I_n = \int_a^b w(x')\, y_n(x')\, y_n(x')\, dx'$

swap Σ and $\int$

$$f(x) = \int_a^b f(x') \sum_{n=0}^{\infty} \frac{y_n(x')\, w(x')\, y_n(x)}{I_n}\, dx'$$

whoa dudes ... this is a sifting character!

oh dear $$\boxed{\sum_{n=0}^{\infty} \frac{y_n(x')\, w(x')\, y_n(x)}{I_n} = \delta(x - x')}$$

↳ <u>completeness relation</u>

clearly just another way aof looking at orthogonality

Eigenvalues are Real

Reality of Eigen function

For $f(x)$, $g(x)$, and $w(x)$ real it is still conceivable to have complex eigenfunctions (e^{ikx} is a solution to $\frac{d^2y}{dx^2}+k^2y=0$)

However for w real ≥ 0 eigenvalues are real

∴ can show this by conjugating SL eqn

$$\frac{d}{dx}\left[f(x)\frac{dy_n^*}{dx}\right]-g(x)\,y_n^*+\lambda_n^*\,w(x)\,y_n^*=0 \quad ①$$

and/or

$$\frac{d}{dx}\left[f(x)\frac{dy_m}{dx}\right]-g(x)\,y_m+\lambda_m\,w(x)\,y_m=0 \quad ②$$

multiply 1 by y_m and 2 by y_n^* and subtract

(term in $g(x)$ cancels)

$$\underset{a}{y_m\frac{d}{dx}\left[f(x)\frac{dy_n^*}{dx}\right]}-\underset{b}{y_n^*\frac{d}{dx}\left[f(x)\frac{dy_m}{dx}\right]}+(\lambda_n^*-\lambda_m)\,w(x)\,y_m\,y_n^*=0$$

integrate over domain → look at first two terms --->

March 19

March 19

Su	Mo	Tu	We	Th	Fr	Sa
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30
31						

April 19

Su	Mo	Tu	We	Th	Fr	Sa
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30				

	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
24/2 - 1/3	Feb-24	25	26	27	28	Mar-1 22	2
3 - 8/3	3	4 23	5	6 24	7	8 25 Ass 3 in	9
10 - 15/3	10	11 26 ~~CWTM~~	12	13 27 MT 2 ~~CWTM~~	14	15 28	16
17 - 22/3	17	18 29	19	20 30	21	22 31 Ass 4	23
24 - 29/3	24	25 32 present	26	27 33 present	28	29 34 Ass 4 in	30
31/3 - 5/4	31	Apr-1 35	2	3 36	4	5 37 ?/	6

$$\left[\overset{a}{\downarrow}\, y_m f(x) \frac{dy_n^*}{dx} - \overset{b}{\downarrow}\, y_n^* f(x) \frac{dy_m}{dx}\right]_a^b = 0 \text{ by b.c.'s}$$

$$+ \int_a^b \left[f(x) \overset{a}{\downarrow}\frac{dy_n^*}{dx} \cancel{\frac{dy_m}{dx}} - f(x) \overset{b}{\downarrow}\frac{dy_m}{dx} \cancel{\frac{dy_n^*}{dx}} \right] dx$$

sign change are =

$$= (\lambda_m - \lambda_n^*) \int_a^b w(x)\, y_n^* y_m \, dx$$

For $m = n$

$$0 = (\lambda_n - \lambda_n^*) \int_a^b w(x)\, y_n^* y_n \, dx$$

$$= (\lambda_n - \lambda_n^*) \underbrace{\int_a^b w(x)\, |y_n|^2 \, dx}_{\geq 0 \text{ for } w(x) \geq 0} = 0$$

/

$\lambda_n = \lambda_n^*$ $\Longrightarrow$ λ most be real !

also get $\int_a^b w(x)\, y_n y_n^* = 0 \quad m \neq n$

(orthogonality for y complex

Consider string

$$\frac{d^2X}{dx^2} + k^2 X = 0, \quad \frac{d^2T}{dt^2} + k^2 v^2 T = 0$$

what if we wanted periodic b.c.'s

$y(0) = y(L)$ $\quad y'(0) = y'(L)$

solution finds two eigenfunctions

$\sin k_n x$, $\cos k_n x$ $\quad k_n = \frac{2n\pi}{L}$

But sin & cos are still orthogonal on interval

And you can construct a new eigen function simply by shifting the origin

$$\sin \frac{2\pi n}{L}(x - x_0) = \sin \frac{2n\pi x}{L} \cos \frac{2n\pi x_0}{L} - \cos \frac{2n\pi x}{L} \sin \frac{2n\pi x_0}{L}$$

linear combination of earlier eigen functions

Degeneracy

What if $\lambda_m = \lambda_n$! FAILure!

This is called degeneracy, if N such eigen functions match N fold degeneracy.

↳ The e-functions "could" still be orthogonal ~~degenerate~~ ← 8a ↓

And, for double degeneracy can construct linear ~~c~~ combinations of new eigen functions that ARE orthogonal.

→ degeneracy reflects symmetry in the physical system

→ In 2 Dimensions

Consider potential in a box...

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + k^2 f = 0$$

Eigen functions are $f_{nm} = \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{W}$

Eigen values $k^2_{nm} = \left(\frac{n^2}{L^2} + \frac{m^2}{W^2}\right)\pi^2$

but if box is square... $L=W$ → Degenerate

Consider Sturm Liouville Operator

Define

$$L\,y \equiv \frac{d}{dx}\left[f(x)\,\frac{dy}{dx}\right] - g(x)\,y$$

So now the SL equation becomes

$$L\,y(x) + \lambda\,w(x)\,y(x) = 0$$

If you use th L operator back in eqn † back on page 02 can write
[thats eqn you mpy y_m & y_n etc]

$$\int_a^b y_n\,L[y_m]\,dx = \int_a^b y_m\,L[y_n]\,dx = 0$$

self-adjoint

Applying SL

$$\frac{d}{dx}\left[f(x)\frac{dy}{dx}\right] - g(x)\,y + \lambda\, w(x)\, y = 0$$

$$w(x) \geq 0 \qquad a \leq x \leq b \qquad \begin{array}{l} \alpha_1 y + \beta_1 y' = 0\big|_a \\ \alpha_2 y + \beta_2 y' = 0\big|_b \end{array}$$

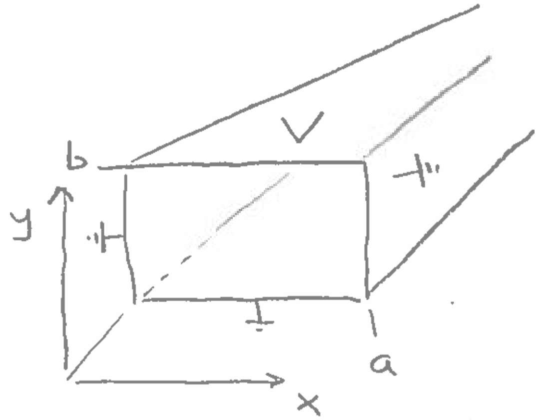
SL equations arise in PDE's that are solved by separation of variables where the two eqns are coupled through the separation constant.

1) Determine set of functions that satisfy each DE
2) Find coordinates with homogeneous b.c.'s
   (ie function or derivative is 0 on BOTH ends) SL requirement.
3) Choose the eigenfunction that satisfies one of the 0 b.c.'s
4) Choose the eigen value that satisfy the second b.c.
5) Repeat 3 & 4 for other coords with Two 0 b.c.'s
6) For the remaining coordinate choose solution that satisfies a 0 b.c. at one boundary.
   The remaining non-0 b.c. identifies the solution
7) General solution is a lin. comb. of eigen functions
8) Use orthogonality to solve for coefficients

hwe end 6/3/19

Google Wave Guide

Example Find electrostatic potential inside an ∞ long rectangular wave guide with conducting walls



One surface is set to voltage V the other surfaces are grounded

Find ϕ in region $0 < x < a$, $0 < y < b$, $\forall z$

$\phi = 0$ $x = 0, a$

$= 0$ $y = 0$

$= V$ $y = b$

Potential satisfies Laplaces eqn

$$\nabla^2 \phi = 0$$

Use separation of variables:

$$\phi(x,y) = X(x)\,Y(y)$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = 0$$

mpy by $1/XY$

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = 0$$

let

$$\boxed{\begin{aligned} \frac{1}{X}\frac{d^2 X}{dx^2} &= -\lambda \\ \frac{1}{Y}\frac{d^2 Y}{dy^2} &= \lambda \end{aligned}}$$

As simple as it gets: Both eqns are SL

$f(x) = 1 \quad g(x) = 0 \quad w(x) = 1$

Boundary Conditions

$X(0) = X(a) = 0 \quad \leftarrow$ SL

$Y(0) = 0, \quad Y(b) = V \leftarrow$ not SL

Follow steps:

~~cont 6/3/77~~

1) Find solutions to DE's

For X $\quad x = X_0 e^{\pm i\sqrt{\lambda}\,x}$

For Y $\quad Y = Y_0 e^{\pm i\sqrt{-\lambda}\;y} = Y_0 e^{\mp\sqrt{\lambda}\,y}$

2) The X eqn has homogeneous b.c.'s

3) Find soln. to X that satisfies one of the b.c.'s

go with

$X \sim \sin\sqrt{\lambda}\,x$

for safety sake $\lambda = k^2 > 0$

$\sin 0 = 0$ so have one b.c.

4) Now find eigenvalue to satisfy other b.c.

need $\sin ka = 0 \Rightarrow ka = n\pi$

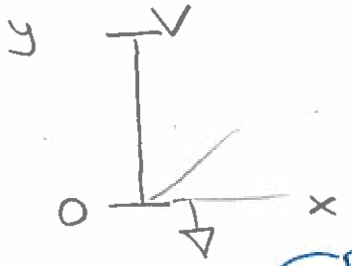
$$\boxed{k = \frac{n\pi}{a}}$$

$X = X_0 \sin\frac{n\pi}{a}x$ ← Eigen function in X

↑ arbitrary amplitude

5) repeat 3,4 for other homogeneous b.c.'s
[None exist here]

6) Choose function to satisfy one b.c.



$$Y = Y_0 e^{\mp\boxed{\frac{n\pi}{a}}y}$$

Separation constant (eigen value)

$$= Y_0 \sinh \frac{n\pi}{a} y$$

$$Y_0 \cosh \frac{n\pi}{a} y$$

at $y=0$, $Y=0$ so

$$Y = Y_0 \sinh \frac{n\pi}{a} y \leftarrow$$

Solution satisfies one b.c.

wave guide x dimension arrives here in separation constant.

7) The general solution is:

$$\phi(x,y) = \sum C_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} y$$

We have one more b.c. and can use to establish C_n

$$\phi(x,b) = \sum C_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a} = V$$

8) Find C_n by using orthogonality of SL eigen functions (in this case sin)

$$\int_0^a V \sin\frac{m\pi x}{a}\,dx = \int_0^a \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a}\,\sinh\frac{n\pi b}{a}\,\sin\frac{m\pi x}{a}\,dx$$

$$-\frac{Va}{m\pi}\cos\frac{m\pi x}{a}\Big|_0^a = \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi b}{a}\underbrace{\int_0^a \sin\frac{n\pi x}{a}\,\sin\frac{m\pi x}{a}\,dx}_{0 \text{ or } m=n\ \frac{a}{2}}$$

$$-\frac{Va}{m\pi}\left(\cos m\pi - 1\right) = $$

$$\sum_{n=1}^{\infty} C_n \sinh\frac{n\pi b}{a}\cdot\frac{a}{2}\,\delta_{nm}$$

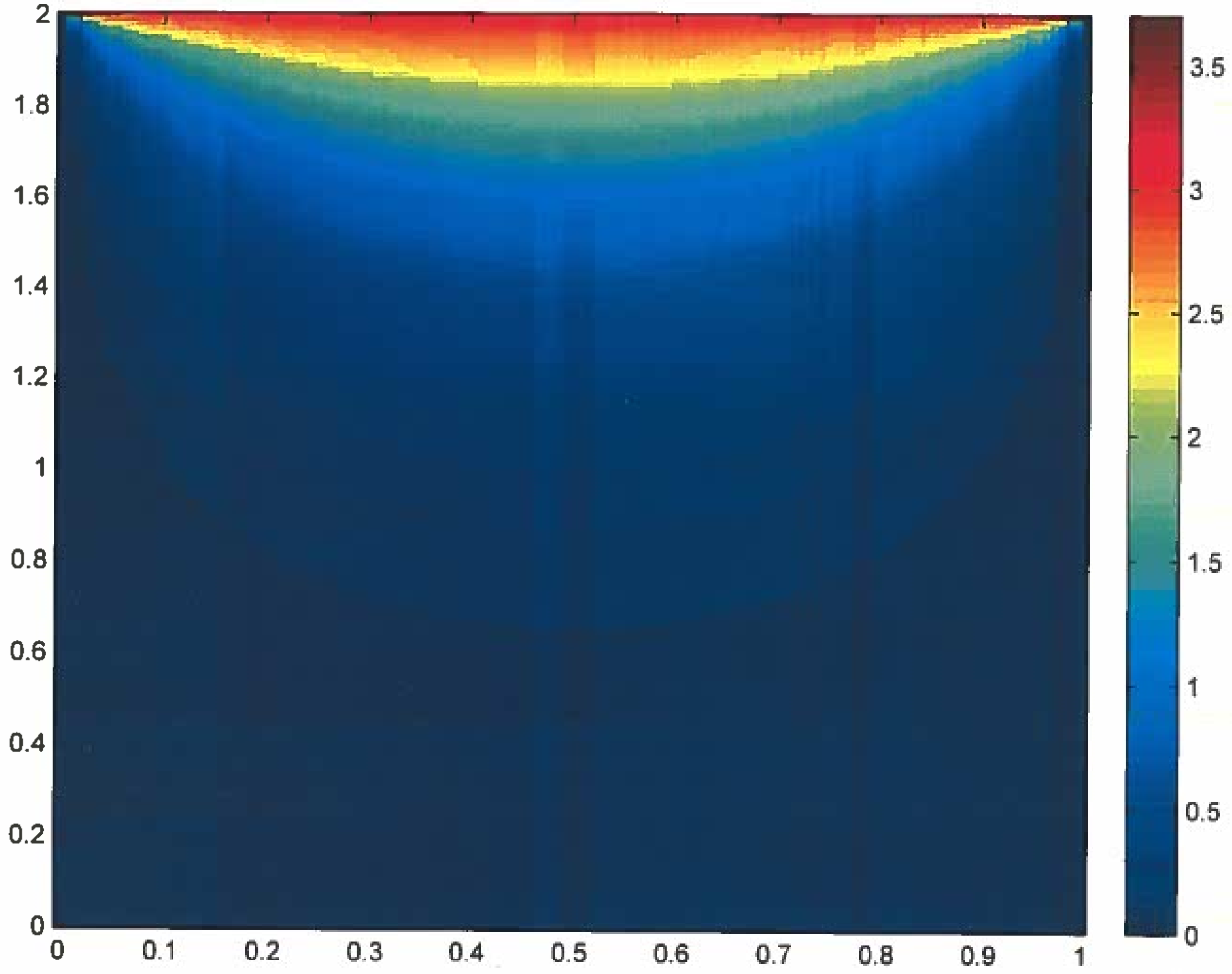
$$\frac{Va}{m\pi}\left[1-(-1)^m\right] = C_m \sinh\frac{m\pi b}{a}\cdot\frac{a}{2}$$

$$C_m = \frac{2V}{m\pi}\,\frac{\left(1-(-1)^m\right)}{\sinh\frac{m\pi b}{a}}$$

$$\boxed{\phi(x,y) = \frac{2V}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,\frac{\left(1-(-1)^n\right)}{\sinh\frac{n\pi b}{a}}\cdot\sin\frac{n\pi x}{a}\,\sinh\frac{n\pi}{a}y}$$

note can simplify sum 'cause only odd values non-zero

```
clear

a = 1;
b = 2;

x = 0:.01:a;

y = 0:.01:b;

phi = zeros(length(y),length(x));
for ix = 1:length(x)
    for iy = 1:length(y)

        n = 1:100;
        %phi(iy,ix) = ...
        s = sum(2*(1 - (-1).^n) ./ (n .* sinh(n*pi*b/a)) .* sin(n*pi*x(ix)/a) .* sinh↙
(n*pi*y(iy)/a));
        phi(iy,ix) = s;
    end
end

figure(1)
clf
imagesc(x,y,phi)
set(gca,'YDir','normal')
colorbar

figure(2)
clf
contour(x,y,phi)
```


2
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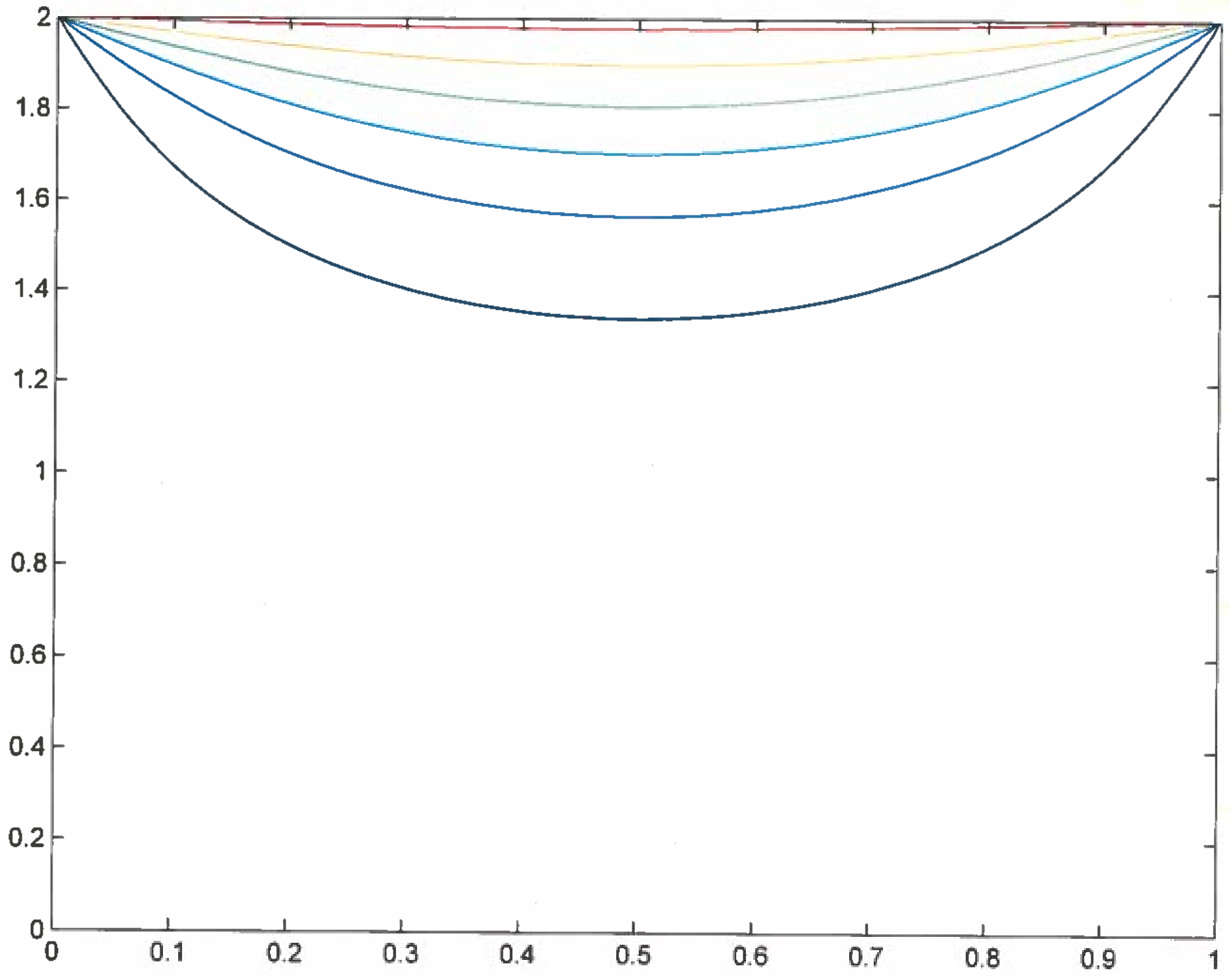
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