Distributions (families of functions)

Start with definitions

Test function: a function $f(x)$ that is infinitely differentiable and $f(x) \to 0$ "fast" as $x \to \infty$ ($f(x) = 0$ for $x > \epsilon$ is safe)

Core functions: $g(x)$ infinitely differentiable

Weak convergence $g_n(x)$ is a sequence of core functions the sequence is weakly convergent if

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} g_n(x) f(x)\, dx$$ exists for any test function $f(x)$

— here end 8/2/19

Distribution

$\phi(x)$ is a distribution if there exists a sequence of core functions $g_n(x)$ that converges weakly to $\phi(x)$

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} g_n(x) f(x)\, dx = \int_{-\infty}^{\infty} \phi(x) f(x)\, dx$$ for any test function

↳ looks like the δ sequence!

More than one series of core functions can converge to the same distribution !

Example

$$g_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}$$

this sequence does not converge pointwise

$g_n(x) \longrightarrow 0$ as $n \to \infty$ $x \neq 0$ good!

But

$g_n(x)$ diverges $n \longrightarrow \infty$ $x = 0$ bad!

But this thing does display weak convergence:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2} f(x)\, dx = \cdots$$

$$\lim_{n\to\infty} \int_{-\infty}^{-1/\sqrt{n}} + \int_{-1/\sqrt{n}}^{1/\sqrt{n}} + \int_{1/\sqrt{n}}^{\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2} f(x)\, dx$$

$$I_1 \quad + \quad I_2 \quad + \quad I_3$$

$\cdot \; \cdot \cdot \; \cdots >$

$$\lim_{n\to\infty} \left(I_1 + I_2 + I_3 \right) = \cdots f(0)$$

and so, the distribution defined by the weak limit of the sequence $g_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$ exhibits the sifting property and is in fact a delta sequence notice that more than one series of core functions can converge to the same distribution!

Example

$$g_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$$

this sequence does not converge pointwise

$g_n(x) \to 0$ as $n \to \infty$ $x \neq 0$ good

but

$g_n(x)$ diverges $n \to \infty$ $x = 0$! bad!

But, this thing does converge weakly! .. look

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2} f(x)\,dx =$$

$$\lim_{n\to\infty} \left(\int_{-\infty}^{-1/\sqrt{n}} + \int_{-1/\sqrt{n}}^{1/\sqrt{n}} + \int_{1/\sqrt{n}}^{\infty} \right) \frac{n}{\sqrt{\pi}} e^{-n^2x^2} f(x)\,dx$$

$$= \lim_{n\to\infty} (I_1 + I_2 + I_3) \underset{\text{transpires}}{\overset{\text{it}}{=}} = f(0)$$ Now skip to p 56

Now the test function is real nice $f(x) \to 0$ or $x \to \pm\infty$

$$|f(x)| \leq M_1 \qquad 0 \leq x \leq \infty \; !$$

so that $I_3 \leq \frac{M_1}{2} \int_{1/\sqrt{n}}^{\infty} \frac{2n}{\sqrt{\pi}} e^{-n^2x^2} f(x)\,dx$

let $u = nx$

$$n\,dx = du$$

$$I_3 = \frac{\Gamma_1}{2}\int_{\sqrt{n}}^{\infty} \frac{2}{\sqrt{\pi}}\, e^{-u^2}\, du$$

$$x = 1/\sqrt{n} \rightarrow u = n \cdot 1/\sqrt{n} = \sqrt{n}$$

Now . . .

$$\Phi(p) \equiv \frac{2}{\sqrt{\pi}}\int_0^p e^{-t^2}\, dt \simeq 1 - \frac{1}{\sqrt{\pi}}\frac{e^{-p^2}}{p} + \dots$$ error function

which incidentally has the property that $\int_0^\infty$

$$\lim_{x\to\infty} \Phi(x) = 1$$

i.e. $$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\, dt = 1$$

Please tell me you've seen the erf

our integral $$\int_{\sqrt{n}}^{\infty} = \int_0^\infty - \int_0^{\sqrt{n}} = 1 - \Phi(\sqrt{n})\ !$$

$$I_3 = \frac{\Gamma_1}{2}\left(1 - \Phi(\sqrt{n})\right)$$

$$\lim_{n\to\infty} I_3 = \underline{\underline{0}}\ !$$

I_1 will suffer the same fate.

29c

$$I_2 = \int_{-1/\sqrt{n}}^{1/\sqrt{n}} \frac{n}{\sqrt{\pi}} e^{-n^2x^2} f(x)\, dx$$

mean value thm

change variable

$$= f(\xi) \cdot \int_{-1/\sqrt{n}}^{1/\sqrt{n}} \frac{n}{\sqrt{\pi}} e^{-n^2x^2}\, dx$$

$u = nx \big|_{x=\frac{1}{\sqrt{n}}} = \sqrt{n}$

even fn'

$$= f(\xi)\, \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{n}} e^{-u^2}\, du$$

$\boxed{\begin{array}{l} u = nx \text{ again} \\ du = n\,dx \end{array}}$

where by ~~using~~ the mean value theorem

$\Phi(x) = \text{erf} = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$

$$-\frac{1}{\sqrt{n}} \leq \xi \leq \frac{1}{\sqrt{n}}$$

Thus as we squeeze this thing in the limit $n \to \infty$

$$I_2 = f(\xi)\, \Phi\sqrt{n}$$

ξ becomes 0 and $\Phi\sqrt{n} \to 1$

$\Phi(\infty) \to 1$

and $I_2 \to f(0)$!

$f(0)$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2} f(x)\, dx = \lim_{n\to\infty} I_2 = \text{“}\int_{-\infty}^{\infty} \phi(x) f(x)\, dx\text{''}$$

21.8/40

30.5

$n = 54$

$n_{SW} = 76.7$

So, the distribution defined as the weak limit of the sequence $g_n(x)$

$= \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$ exhibits the sifting property and is in fact

the delta ~~function~~ sequence!

$\frac{n}{\sqrt{\pi}} e^{-n^2x^2} \to \delta(x)$ (weakly)

So we have a new kind of convergence. Mathematical convergence is very strong, it regulates behaviour at every point. Physically you can't measure things at a point anyway and so weak convergence is okay.

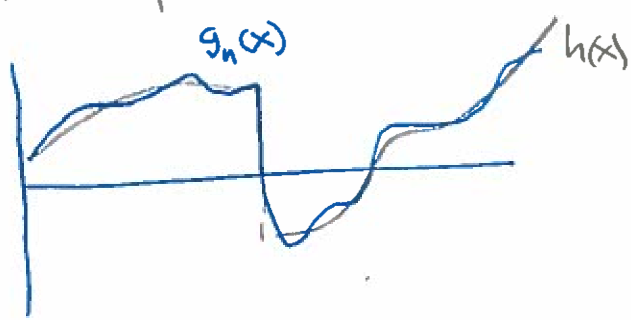
The smudging theorem

Not differentiable

For every continuous function $h(x)$ we can construct a sequence $g_n(x)$ of core functions such that

$$\lim_{n \to \infty} \left| g_n(x) - h(x) \right| < \epsilon$$

notice $\frac{dh}{dx} = \infty$



for any $\epsilon > 0$ $\forall\, x$ on a finite interval

English you can always smooth the function $h(x)$ over the interval $\left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ creating an infinitely differentiable (ie smooth) $g_n(x)$ so that $g_n(x)$ and $h(x)$ differ negligibly ($< \epsilon$)

You can therefore replace any function with a distribution!

> For every continuous function, an equivalent distribution exists

end 11/2/19

(The opposite is not in general true)

↳ there is no continuous function corresponding to the delta function.

→ The delta function is NOT a function but a distribution ←

Properties of distributions

① Distributions can be added, subtracted, multiplied by constants or multiplied by infinitely differentiable functions.

most are obvious but... what about differentiable fn.?

→ $h(x)$ infinitely differentiable function

$\phi(x)$ the limit of the sequence $g_n(x)$

↳ (∞ differentiable)

$$\int_{-\infty}^{\infty} [h(x)\,\phi(x)]\, f(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} [h(x)\, g_n(x)]\, f(x)\, dx$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)\, [h(x)\, f(x)]\, dx = \int_{-\infty}^{\infty} \phi(x)\, \underbrace{h(x)\, f(x)}_{\text{will work as test fn!}}\, dx$$

Note you can't casually manipulate the distribution as if it were a function it's not

But the sequence ... thats cool.

So ... if $\phi(x)$ was a distubution so to is $h(x) \cdot \phi(x)$!

Example $x \cdot \delta(x)$

$$\int_{-\infty}^{\infty} [x\, \delta(x)]\, f(x)\, dx = \int_{-\infty}^{\infty} \delta(x)\, [x \cdot f(x)]\, dx$$

$$= 0 \cdot f(0) = 0\,!$$

(not as useless as it looks)

2. You may not multiply / divide distributions by other distributions

3. if $\phi(x)$ is a distribution so also are $\phi(x-a)$, $\phi(ax)$

4. Distributions are infinitely differentiable (because the core functions are!)

If $\quad g_n(x) \rightarrow \phi(x)$ weakly

$$g_n'(x) \rightarrow \phi'(x) \quad !$$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n'(x)\, f(x)\, dx = \lim_{n\to\infty} \Big\{ \underbrace{f(x)\, g_n(x) \Big|_{-\infty}^{\infty}}_{\to 0 \text{ on strength of test function}} - \int_{-\infty}^{\infty} g_n(x)\, f'(x)\, dx \Big\}$$

$$= -\int_{-\infty}^{\infty} \phi(x)\, f'(x)\, dx$$

$$\int_{-\infty}^{\infty} \phi'(x)\, f(x)\, dx = -\int_{-\infty}^{\infty} \phi(x)\, f'(x)\, dx$$

(similar to δ fn result)

Okay -- if distribution can be differentiated ... it can be a solution to a differential eqn!

<u>like</u> $\qquad (x-1)\dfrac{dy}{dx} + y = 0$

$$\frac{dy}{dx} = \frac{-y}{x-1} \implies \frac{-dy}{y} = \frac{dx}{x-1}$$

$$-\ln y = \ln(x-1) + A$$

$$e^{-\ln y} = e^{\ln(x-1) + A}$$

$$\frac{1}{y} = B(x-1)$$

A, B, and C are just some (related) constants

$$\boxed{y = \frac{C}{(x-1)}}$$

solution works for $x \neq 1$

Now try

$$y = \delta(x-1)$$

$$(x-1)\cdot\frac{dy}{dx} + y = 0$$

$$\underbrace{\int_{-\infty}^{\infty}\Big[\underbrace{(x-1)\,\delta'(x-1)}_{A} + \underbrace{\delta(x-1)}_{B}\Big]\cdot f(x)\,dx}_{} \stackrel{?}{=} 0$$

need to use weak convergence concept integral

$$A \Rightarrow \int_{-\infty}^{\infty}(x-1)\,\delta'(x-1)\,f(x)\,dx = -\int_{-\infty}^{\infty}\delta(x-1)\frac{d}{dx}\big[(x-1)\cdot f(x)\big]\,dx$$

recall $\left(\int \phi' f(x) = -\int \phi\, f'(x)\right)$

$$-\int_{-\infty}^{\infty}\delta(x-1)\big[(x-1)f'(x) + f(x)\big]\,dx$$

$$A = -f(1)$$

And the second part

$$B \Rightarrow \int_{-\infty}^{\infty}\delta(x-1)\,f(x)\,dx = f(1)$$

$$A+B = -f(1) + f(1) = \underline{\underline{0}}$$

So ... in a distribution sense the equation is satisfied and it includes the point $x=1$ in domain!

Some de's CONTAIN distributions. The solutions to these de's are also distributions

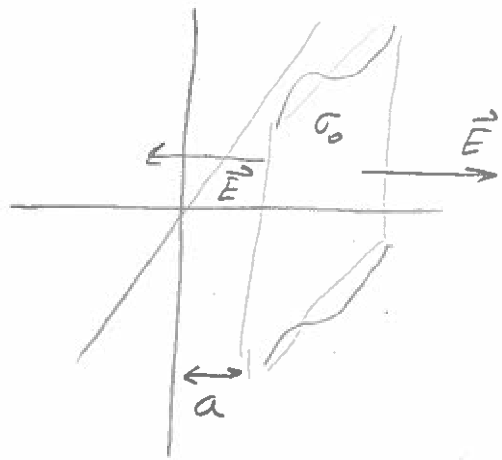
look at $\rightarrow$ $\frac{dE_x}{dx} = \frac{\sigma_0}{\epsilon_0}\,\delta(x-a)$

Find solution to Poisson's equ for the electric field of a sheet of charge with charge density

$$\rho_0 = \sigma_0 \, \delta(x-a)$$

$$\nabla^2 \phi = \frac{-\rho}{\epsilon_0} \qquad \vec{E} = -\nabla \phi \qquad \phi \text{ is electric potential}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$



gives

$$\frac{dE_x}{dx} = \frac{\sigma_0}{\epsilon_0} \delta(x-a)$$

$$\boxed{\delta(x) = \frac{d}{dx} \Theta(x)}$$

δ function step function

$$\frac{dE_x}{dx} = \frac{\sigma_0}{\epsilon_0} \frac{d}{dx} \Theta(x-a)$$

$$E_x = \frac{\sigma_0}{\epsilon_0} \Theta(x-a) + C$$ important!

we know E points away from sheet symmetrically so $E(a-s) = -E(a+s)$

$$\frac{\sigma_0}{\epsilon_0} \underbrace{\cancel{\Theta(-s)}}_{=0!} + C \;\; \overset{x=a-s \qquad x=a+s}{=} \;\; -\left[\frac{\sigma_0}{\epsilon_0} \underbrace{\Theta(s)}_{=1} + c\right]$$

$$C = -\frac{\sigma_0}{\epsilon_0} - C$$

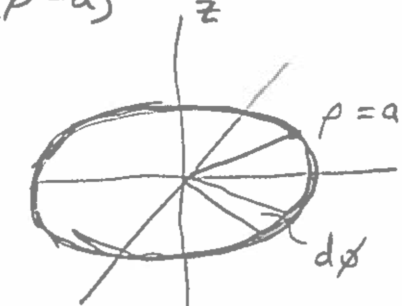
$$C = -\sigma_0/2\epsilon_0$$

$$\boxed{E_x = \frac{\sigma_0}{\epsilon_0}\left[\Theta(x-a) - \frac{1}{2}\right]}$$

Describing physical quantities with delta functions

Consider a $\boxed{\text{ring}}$ of charge of radius a in the x-y plane with charge line density $\lambda_0 \cos\phi$ what is the volume charge density in polar & cylindrical coordinates?

$$\rho(\vec{x}) = C(\rho, z)\, \delta(z)\, \delta(\rho - a)$$



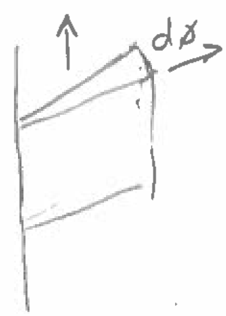
To find C you need to $\int$ charge density over some infinitesimal region for which we know the answer...

Full range in z and ρ but only small known bit in ϕ

choose $\int_{z\,0}^{\infty} \int_{\rho\,0}^{\infty}$ but infinitesimal in ϕ

No δ

This is an infinite wedge



$$\left(\int_{d\phi}\right) \int_z \int_\rho \rho(\vec{x}) = \underbrace{a\, d\phi}_{\text{length along charge ring}}\ \underbrace{\lambda_0 \cos\phi}_{\text{charge density}}$$

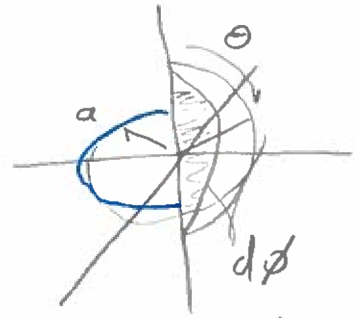
$$= \iiint_{z\,\rho\,\phi} C(\rho, z)\, \delta(z)\, \delta(\rho - a)\, \rho\, d\rho\, dz\, d\phi$$

$$a \lambda_0 \cos\phi\, d\phi = a\, C(\rho - a, 0)\, d\phi \implies C = \lambda_0 \cos\phi$$

shoulda guessed that one

more interesting in spherical coordinates …

$$\rho(\vec{x}) = C\ \delta\left(\theta - \tfrac{\pi}{2}\right)\ \delta(r-a)$$



Now our integral

$$a\,\lambda_0 \cos\phi\, d\phi = \int_{\text{wedge}} C\ \delta\left(\theta - \tfrac{\pi}{2}\right)\delta(r-a)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Wedge
θ $0 \to \pi$
r $0 \to \infty$ — but just section $d\phi$

$$a\,\lambda_0 \cos\phi\, d\phi = C\, a^2 \underbrace{\sin\tfrac{\pi}{2}}_{1}\, d\phi$$

$$C = \frac{\lambda_0}{a}\cos\phi$$

$$\boxed{\rho(\vec{x}) = \lambda_0 \cos\phi\ \delta\left(\theta - \tfrac{\pi}{2}\right)\frac{\delta(r-a)}{a}}$$

move

if you integrate over full volume could solve for C
BUT you'd lose the ϕ dependence

Again in general

1. find all coord that take on single value $\Rightarrow$ δ's
2. determine a fn that multiplies the δ's that goes full range on the coordinates of the δ but infinitesimal else $\Rightarrow$ find this value trivially and it must agree with $\iiint_\delta$ the (restricted) volume integral

Green's function is a distribution that describes the response of a physical system to a unit ~~impulse~~ delta function input. The response of the LINEAR system to a more complicated force/input can be found by integrating over a collection δ inputs to build up the response to an arbitrary input.

For example the ~~input~~ impulse input/unit length of a suspended string is:

Start at END for Rob

$$I(x) = \int_0^L I(x')\,\delta(x-x')\,dx'$$

impulse/length

magnitude input function at any x'

collection of δ's

If we know the response of the system to $\delta(x-x')$ call that $G(x,x')$ then

$$y(x) = \int_0^L I(x')\,G(x,x')\,dx$$

input at x' can cause output $\forall$ x values in general $G(x, x') = G(x', x)$

and if a time variable

$$G(t, t') = G(-t', -t)$$

forced by causality

Three possible approaches:

1) Division-of-region
2) Eigen value — not doing right now
3) integral transform — maybe ... it's very cute

1) - Write differential eqn without sources $\Rightarrow$ homogeneous equation

- Now write the problem for the Green's function and now the forcing will be the δ where for now location is fixed a $x = x' \Rightarrow \delta(x - x')$

- Hence division of region $x < x'$ and $x > x'$

- Find appropriate solutions to match b.c.'s in two regions $\Rightarrow$ 1 or more constants introduced

- will also have created a b.c. at $x = x'$

- Use $G(x, x')$ to find response to any input.

Consider particle acted on by force with a damping term that varies $\propto v$

$$m\frac{dv}{dt} + \alpha v = F(t)$$

Boundary conditions:

$$v \to 0 \quad \text{as} \quad t \to \pm\infty$$

Equation for Green's function:

replaced $F(t)$ with δ

$$m\frac{dG}{dt} + \alpha G = \delta(t-t')$$

For $t \neq t'$

$$m\frac{dG}{dt} + \alpha G = 0$$

$$G = A e^{-\alpha/m\, t}$$

$G \to 0$ as $t \to \infty$ good

$G \to \infty$ as $t \to -\infty$ bad $\quad t > t'$ and $G=0 \quad t<t'$

system is set in motion at $t = t'$

$$G(t,t') = \begin{cases} 0 & t<t' \\ A e^{-\alpha t/m} & t>t' \end{cases}$$

what's missing on rhs? .... t'

t' must lurk in A!

Need to find A

The δ input occurs at $t = t'$ thereby dividing the domain. Integrate equation across boundary:

$$\underbrace{\int_{t'-\epsilon}^{t'+\epsilon} \left[m \frac{dG}{dt} + \alpha G\right] dt}_{} = \underbrace{\int_{t'-\epsilon}^{t'+\epsilon} \delta(t-t')\, dt = 1}_{\text{We know this}}$$

$$\underbrace{m G \Big|_{t'-\epsilon}^{t'+\epsilon}}_{A} + \alpha \underbrace{\int_{t'-\epsilon}^{t'+\epsilon} G\, dt}_{B} = 1$$

$$B = \left| \int_{t'-\epsilon}^{t'+\epsilon} G\, dt \right| \leq \max|G|\,(2\epsilon) \leq \underline{\underline{2\epsilon |A|}}$$

$$G = A \underbrace{e^{-\alpha t/m}}_{\leq 1}$$

integral $\to 0$ as $\epsilon \to 0$!

A at the lower limit $t < t' \Rightarrow G = 0$

$t > t' \Rightarrow G = A e^{-\alpha t/m}$

$$\lim_{\epsilon \to 0} G = m\left[A e^{-\alpha t'/m} - 0\right] = 1$$

$$\boxed{G(t,t') = \begin{cases} 0 & t < t' \\ \frac{1}{m} e^{-\frac{\alpha}{m}(t-t')} & t > t' \end{cases}}$$

Piecewise solution always results from division of region approach. Notice time reciprocity property apparent!

Now we have $G(t,t')$ ... use it

top p499 book

Greens Function Example

$$\underbrace{m \frac{dv}{dt} + \alpha v}= F(t)$$

particle with
damped motion

$u \to 0$ as $t \to \pm\infty$

For the Greens function:

$$m \frac{dG}{dt} + \alpha G = \delta(t-t')$$

and for $t \neq t'$ this is

$$m \frac{dG}{dt} + \alpha G = 0$$

$$G = A e^{-\frac{\alpha}{m} t}$$

this goes to 0 as $t \to \infty$ but $\to \infty$ as $t \to -\infty$ ... BUT nothing happens until $\delta(t-t')$ does its thing so:

$$G(t,t') = \begin{cases} 0 & t < t' \\ Ae^{-\frac{\alpha}{m} t} & t > t' \end{cases}$$

must be $A(t')$ in general

The δ divides the process into two domains

$$\int_{t'-\epsilon}^{t'+\epsilon}\left(m\frac{dG}{dt}+\alpha G\right)dt = \int_{t'-\epsilon}^{t'+\epsilon}\delta(t'-t)\,dt = 1$$

$$mG\Big|_{t'-\epsilon}^{t'+\epsilon} + \alpha\underbrace{\int_{t'-\epsilon}^{t'+\epsilon} G\,dt}\; = 1$$

$\hookrightarrow$ Consider $\left|\int_{t'-\epsilon}^{t'+\epsilon} G\,dt\right| \le \max|G|\, 2\epsilon \le 2\epsilon|A|$

for finite G, this $\int \to 0$ as $\epsilon \to 0$ <u>be gone</u>

Now go to the first term

for $G\big|_{t'-\epsilon}$ $\quad t<t' \quad G=0$

for $G\big|_{t'+\epsilon}$ $\quad t>t' \quad G = Ae^{-\alpha t/m}$

so as $\epsilon \to 0$

$$m\left[Ae^{-\alpha t'/m} - 0\right] = 1$$

$$\Rightarrow \quad A(t') = \frac{1}{m}e^{\frac{\alpha}{m}t'}$$

AND

$$G(t,t') = \begin{cases} 0 & t<t' \\ \frac{1}{m} e^{-\frac{\alpha}{m}(t-t')} & t>t' \end{cases}$$ (2 domains)

Piecewise definition characteristic of the "division-of-region" method used here.

Consider $F(t) \rightarrow \begin{cases} 0 & t<0 \\ F_0 & t>0,\ t<T \\ 0 & t>T \end{cases}$ (3 domains)

Use Greens function ~~for t<0 v(t)=0~~ ! obvious nothing yet happened!

$$v(t) = \int_{-\infty}^{\infty} F(t')\, G(t,t')\, dt'$$

end 27/2/19

$$= \int_{-\infty}^{0} \underset{t'<0}{0}\ G(t,t') + \int_{t'=0}^{T} F_0\ G(t,t') + \int_{t'=T}^{\infty} 0\ G\, dt'$$

0 again

~~for t>0~~

for $t>0$ $\quad v(t) = F_0 \int_{t'=0}^{T} G(t,t')\, dt'$

~~For 0<t<T~~

~~(v(t)<0 = 0)~~

for $t<0$

$v(t) = 0$

limits on this $\int$ are a bit funny because $G = 0$ for $t < t'$

$$v(t) = F_0 \int_0^T \frac{1}{m} e^{-\alpha/m (t-t')} dt'$$

$t' > t$, $t - t' < 0$ $G = 0$!

$\underline{\underline{t' > t \rightarrow 0}}$

$$= F_0 \int_0^t \frac{1}{m} e^{-\frac{\alpha}{m}(t-t')} dt'$$

$- \cdot - = +$

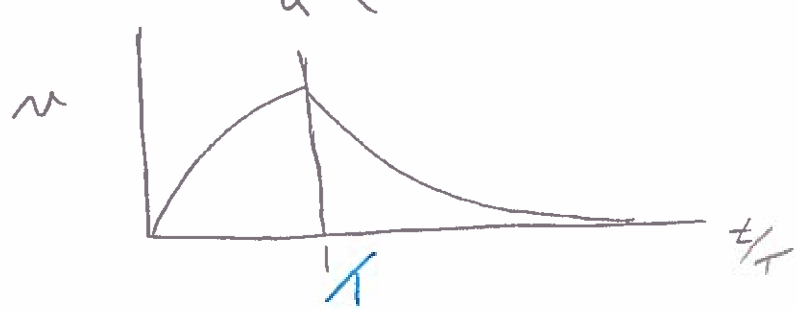
$$= + \frac{F_0}{m} \frac{1}{\alpha/m} e^{-\frac{\alpha}{m}(t-t')} \Big|_0^t = \frac{F_0}{\alpha}\left(1 - e^{-\frac{\alpha}{m} t}\right)$$

$t > T$ $\Rightarrow$ implies $t - t' > 0$ G never 0

$$v(t) = F_0 \int_{t'=0}^T \frac{1}{m} e^{-\alpha/m (t-t')} dt'$$

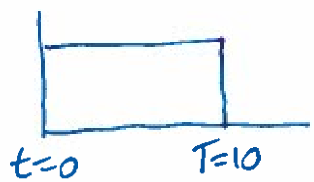
$$= \frac{F_0}{m} \frac{1}{\alpha/m} e^{-\frac{\alpha}{m}(t-t')} \Big|_0^T$$

$$= \frac{F}{\alpha}\left(e^{-\frac{\alpha}{m}(t-T)} - e^{-\frac{\alpha}{m} t}\right)$$



```
clear

m = 1;  % mass

alp = 1;  % damping

% gt = 1/m * exp(-alp/m*(t - tp));

% F = 0 t < 0
%   = Fa t > 0 t < T
%   = =  t > T

T = 10;  % some time relavant to forcing
Fa = 1;  % scale of forcing

tmax = 20;

t = 0:.01:tmax;
FF = zeros(size(t));
ins = find(t<T);
FF(ins) = Fa * ones(size(ins));
```

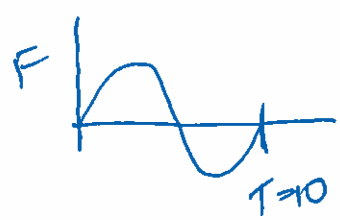
— Forcing




```
tp = 0:.01:tmax;

dt = mean(diff(t));
for ii = 1:length(t)   % ii counts through t
    Gttp = zeros(size(tp));
    ins = find(tp < t(ii));


    Gttp(ins) = 1/m * exp(-alp./m.*(t(ii)-tp(ins)));
```

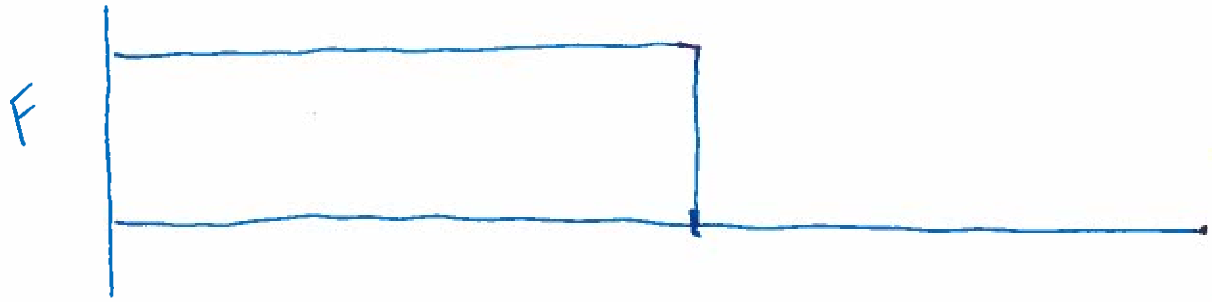
$$\int_0^t F\cdot\frac{1}{m}\, e^{-\frac{\alpha}{m}(t-t')}\, dt$$

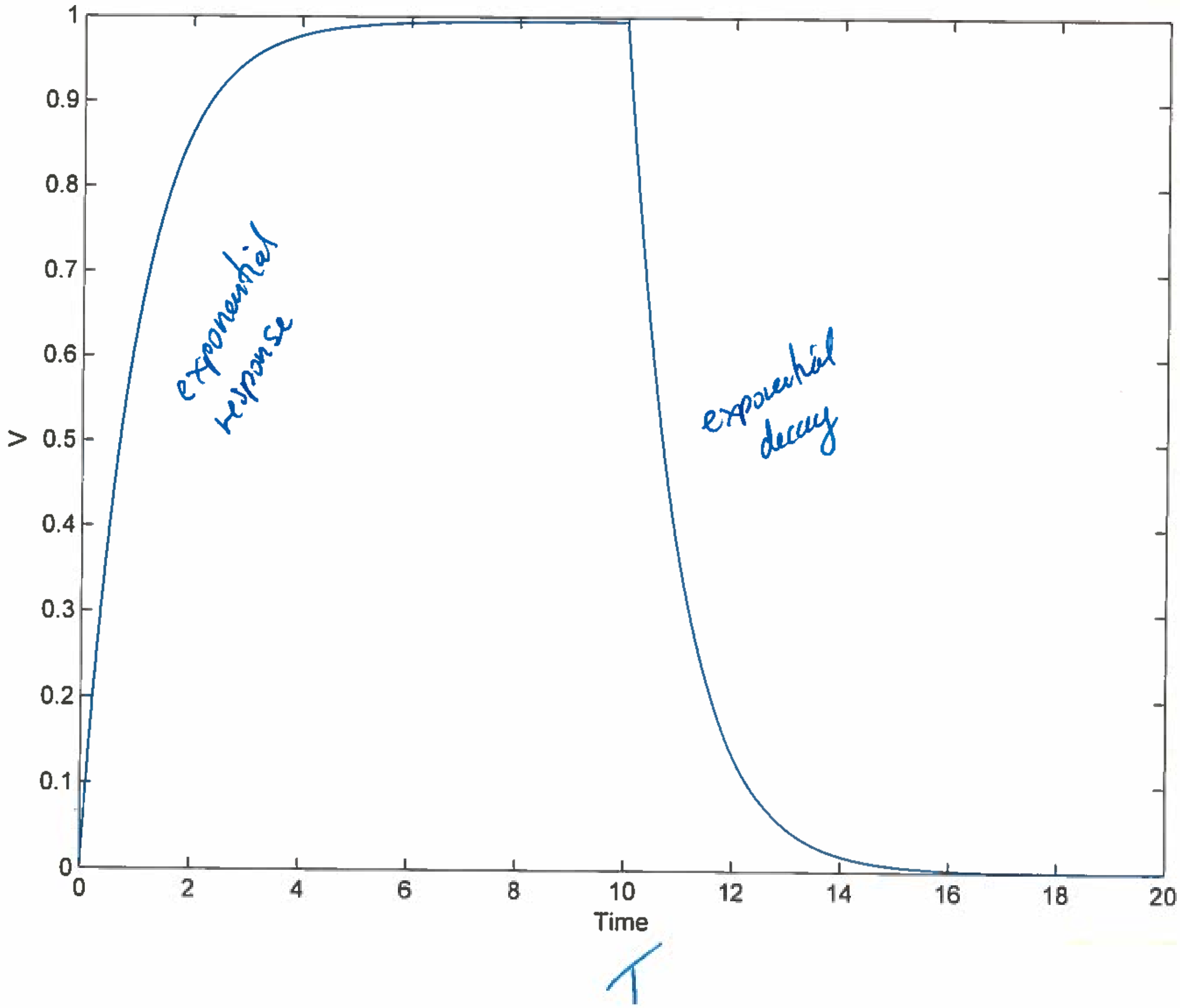
```
    V(ii) = sum(FF .* Gttp)*dt;
end

plot(t,V)
xlabel('Time')
ylabel('V')

ins = find(t<T);
FF(ins) = sin(2*pi/10*t(ins));   % w = .1 gives less than one cycle

figure(2)
clf
subplot(211)
plot(t,FF)
ylabel('Force')
```

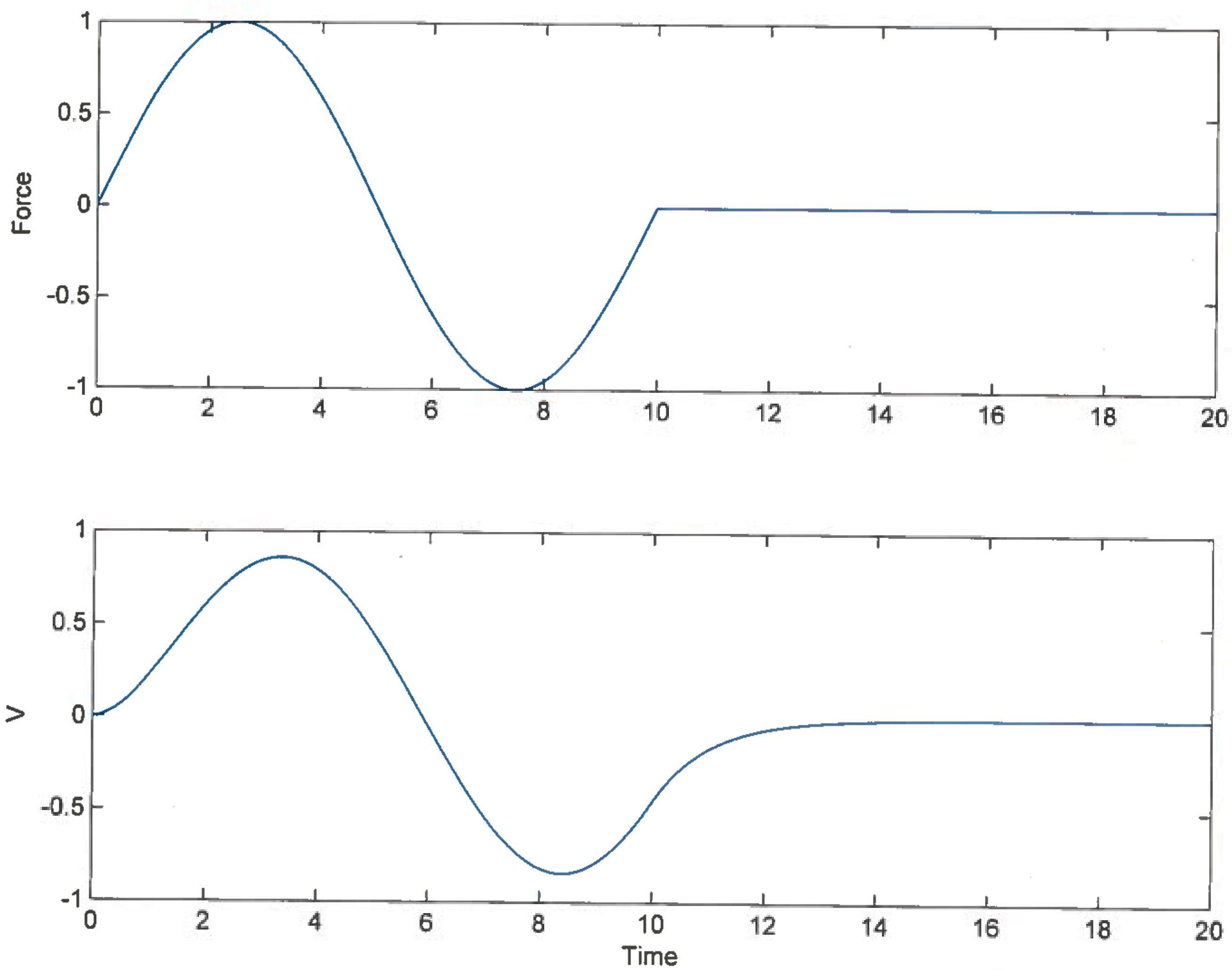



```
for ii = 1:length(t)   % ii counts through t
    Gttp = zeros(size(tp));
    ins = find(tp < t(ii));
```

```
    Gttp(ins) = 1/m * exp(-alp./m.*(t(ii)-tp(ins)));

    V(ii) = sum(FF .* Gttp)*dt;
end

subplot(212)
plot(t,V)
xlabel('Time')
ylabel('V')
```


F


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
V
0
2
4
6
8
10
12
14
16
18
20
Time
exponential response
exponential decay
T



Force
1
0.5
0
-0.5
-1
0
2
4
6
8
10
12
14
16
18
20
V
1
0.5
0
-0.5
-1
0
2
4
6
8
10
12
14
16
18
20
Time