Generalised Functions

→ Might do Fourier Series first if not seen.

We often conceive of point objects or particles ~~and~~ but on occasions we must manipulate mathematical representations of this concept what to do?

We need a function.

$$\delta(t) = 0 \quad \forall\ t \neq 0$$
$$\delta(t) = \infty \qquad t = 0$$
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

this is the ~~δ~~ Dirac delta function.

⇒ for example an impulsive force

$$F(t) = \vec{I}\ \delta(t)$$

$$\vec{I} = \int_{-\infty}^{\infty} \vec{F}(t)\, dt = \vec{I} \int_{-\infty}^{\infty} \delta(t)\, dt = \vec{I} \quad !$$

Similarly you could have a point particle at the origin with the density ~~mass~~ distribution

$$\rho(\vec{r}) = M\ \delta(\vec{r}) = M\ \delta(x)\,\delta(y)\,\delta(z)$$

notice unit funnyies

The particle mass is then

$$M = \iiint_{-\infty}^{\infty} \rho(\vec{r})\, dV = M \iiint \delta(x)\,\delta(y)\,\delta(z)\, dx\, dy\, dz$$

$$= M \ !$$

The dimensions of δ are the inverse of its argument

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \Rightarrow \dim \delta(t) = \frac{1}{T}$$

↳ We are going to need a creative solution to generate this function!

Delta Sequences

we need a mathematical function that provides a "sifting property"

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0) \quad !$$

↳ the δ function "sifts" that value only at $\delta_{x=0}$ all other vanish

you can get an idea of how this works:

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = \int_{-\varepsilon}^{\varepsilon} f(x)\, \delta(x)\, dx \qquad \delta(x) = 0 \quad x \neq 0$$

in limit as $\varepsilon \to 0$

$$= f(x_0) \int_{-\varepsilon}^{\varepsilon} \delta(x)\, dx \qquad -\varepsilon < x_0 < \varepsilon$$

$\boxed{\text{mean value theorem}}$

$$= f(x_0) \int_{-\infty}^{\infty} \delta(x)\, dx$$

$$\int_a^b f(x) g(x)\, dx = f(c) \int_a^b g(x)\, dx$$

for some c $\quad a < c < b$

$$= f(x_0) \cdot 1 \longrightarrow f(0) \text{ as } \varepsilon \to 0$$

you can imagine $\delta(x)$ as the limit of a set of functions concentrated in some band $\pm\varepsilon$ so that you just choose ε as small as it needs to be.

So imagine a series of $\phi_n(x)$ defined in some way such that $\varepsilon \to 0$ as $x \to \infty$ and

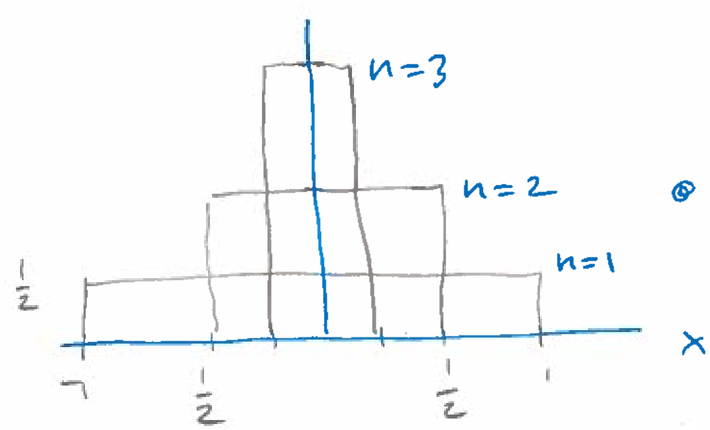
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x)\, f(x)\, dx = f(0)$$

(for linear $f(x)$)

The series $\phi_n(x)$ is called a <u>delta sequence</u>

The simplest delta sequence is a box function:

$$\phi_n(x) = \begin{cases} n/2 & -1/n < x < 1/n \\ 0 & \text{otherwise} \end{cases}$$



Does our function sift

$$\int_{-\infty}^{\infty} \phi_n(x) f(x)\, dx = \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} f(x)\, dx$$

mean value thm
$$= \frac{n}{2} f(x_0) \frac{2}{n} \qquad -\frac{1}{n} < x_0 < \frac{1}{n}$$

$$= f(x_0) \longrightarrow f(0) \quad \text{as} \quad n \to \infty$$

Success of the sifting property demonstrates that ϕ_n is a delta sequence

what about $\delta(x-a)$?

$$\int_{-\infty}^{\infty} \phi_n(x-a) f(x)\, dx = \int_{-\infty}^{\infty} \phi_n(u) f(u+a)\, du$$

$u = x - a$

$$= \frac{n}{2} f(u_0 + a) \cdot \frac{2}{n} \qquad -\frac{1}{n} \leq u_0 \leq \frac{1}{n}$$

$$= f(a) \quad \text{as} \quad n \to \infty$$

can also explore $\delta(ax)$

$$\int_{-\infty}^{\infty} \phi_n(ax) f(x)\, dx = \int_{-\infty}^{\infty} \phi_n(u) f\left(\frac{u}{a}\right) \frac{du}{a}$$

$u = ax$

$$= \frac{n}{2} \frac{2}{n} \frac{1}{a} f\left(\frac{u}{a}\right) \longrightarrow \boxed{\frac{1}{a} f(0)} \quad \text{as } n \to \infty$$

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Notice, if $a<0$ the limits invert but everything else stays the same

$$= -\frac{1}{a} f(0) \quad \text{as } n\to\infty$$

But as $a<0$ $\longrightarrow$ $\boxed{\delta(ax) = \frac{1}{|a|}\delta(x)}$ * watch out for this one

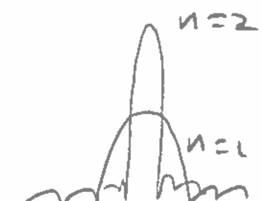
end 25/1/19

The block function is nice but it sucks if you need to take a derivative :

CSC 170 : 171 Quantum Senior Lab

Try

$$\phi_n(x) = \frac{1}{n\pi}\frac{\sin^2 nx}{x^2}$$



$x=0$? $\frac{0}{0}$? ... no in limit it's $\frac{x^2}{x^2}=1$

is infinitely differentiable and meets the peaky character as $n\to\infty$

$$\frac{d\phi_n}{dx} = \frac{1}{n\pi}\left[\frac{2n\sin nx\cos nx}{x^2} - \frac{2\sin^2 nx}{x^3}\right]$$

no problems $x\neq 0$ ... what about small x?

$$\boxed{\left.\frac{d\phi_n}{dx}\right|_{x=0} = \lim_{x\to 0}\left\{\frac{2\sin nx}{n\pi x^3}\cdot\left(nx\cos nx - \sin nx\right)\right\}}$$

Leave up

$\sin nx \simeq nx$ $\quad\cos nx \simeq 1-\frac{n^2x^2}{2}$ ...

restrict to 1st order

so as refer to (it) later.

$$= \lim_{x\to 0} \frac{2nx}{n\pi x^3}\left(nx\left(1-\frac{n^2x^2}{2}\right) - nx\right)$$

$$\frac{2nx}{n\pi x^3}(nx - nx)$$ not good

go to higher order

$$\sin nx = nx - \frac{n^3x^3}{6} \qquad \cos nx = 1 - \frac{n^2x^2}{2}$$

$$\frac{2nx}{n\pi x^3}\left\{\left[\overset{nx\cdot\cos}{nx\left(1-\frac{n^2x^2}{2}+\right)} - \overset{\sin}{\left(nx - \frac{n^3x^3}{6}\right)}\right] - \frac{n^3x^3}{6}\left[\cdots\right.\right.$$

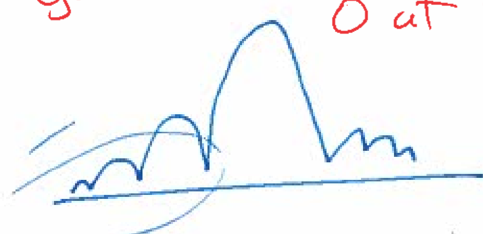
$$\left.\left[\cancel{nx} - \frac{n^3x^3}{2} - \cancel{nx} + \frac{n^3x^3}{6}\right] - \text{hot}\right\}$$

first order

$$\frac{\cancel{2}}{\cancel{n}\pi x^3}\cancel{nx}\cdot -\frac{2n^3\cancel{x^3}}{6} = \frac{-2}{3}\frac{n^3}{\pi}x$$

$$\lim_{x\to 0} -\frac{2}{3}\frac{n^3}{\pi}x \longrightarrow 0$$ for bounded n!!

yes! that's right, slope is 0 at $x=0$!

$\frac{\sin^2 x}{x^2}$ looks like 

side lobes may be a problem.

? Do we know Cauchy Principle Value Thm ?

yes ⋮ no goto ④ page ⑧

Okay what about the sifting property?

$$\int_{-\infty}^{\infty} \frac{1}{n\pi} \frac{\sin^2 nx}{x^2} \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{n\pi x^2} \left(\frac{e^{inx} - e^{-inx}}{2i} \right)^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{n\pi x^2} \left(\frac{e^{2inx} + e^{-2inx} - 2}{-4} \right) f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{+1}{4n\pi} \left(\frac{2 - e^{2inx} - e^{-2inx}}{x^2} \right) f(x)\, dx$$

put on board before class

$$= \frac{1}{4n\pi} \left[\underbrace{\int_{-\infty}^{\infty} \frac{(1 - e^{2inx})}{x^2} f(x)\, dx}_{I1} + \underbrace{\int_{-\infty}^{\infty} \frac{1 - e^{-2inx}}{x^2} f(x)\, dx}_{I2} \right]$$

Evaluate in the complex plane closing the integral path along the x-axis. We'll "suppose" that $f(x) \to f(z)$ is analytic with isolated simple poles AND as usual $f(z) \to 0$ as $z \to \infty$

$\pm$ for $I1$ & $I2$

let $g_n^{\pm}(z) \equiv \dfrac{1 - e^{\pm 2inz}}{z^2}$ — we have a pole at $z=0$!

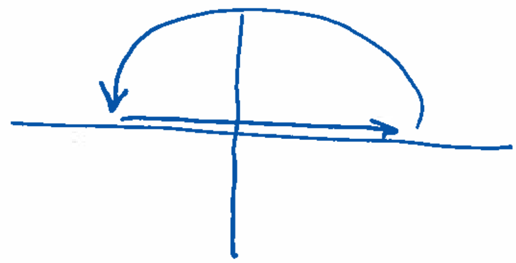
expand as Laurent Series $= \dfrac{1 - \left[1 \pm 2inz - \dfrac{4n^2z^2}{2} \dots\right]}{z^2}$

28/1/19

$$g_n^{\pm}(z) = \mp \frac{2in}{z} + 2n^2 + \cdots$$

simple pole at $z=0$ residue $\mp 2in$

We need $\int_{-\infty}^{\infty}$ so the usual plan would be to go over the top (bottom for I_2) half space



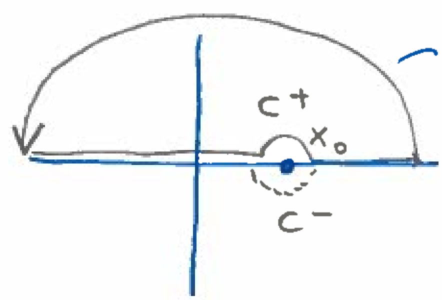
✱ go middle p 10

But we now have to go THROUGH a pole!

[I didn't like Lea's approach here -- she shifts ~~the~~ the integral up a little for one $\int$ ~~and~~ the pole is excluded and for the other it is included ... I don't like that

✱ Arfken old p458 (Cauchy Principle Value)
now p 514 But new analysis!

Consider $f(x)$ with pole on x-axis



go ccw

$\int_{-\infty}^{\infty} f(x)\,dx$ → we introduce small detour around $x=x_0$ → $z-x_0 = \delta e^{i\varphi}$

$dz = i\delta e^{i\varphi}\,d\varphi$

1

So this gives

ccw below
$$\int_{C_-} a_{-1} \frac{dz}{z-x_0} = a_{-1}\, i \int_{\pi}^{2\pi} \frac{\delta e^{i\phi}\, d\phi}{\delta e^{i\phi}} = i\pi a_{-1}$$

cw above
$$\int_{C_+} a_{-1} \frac{dz}{z-x_0} = a_{-1}\, i \int_{\pi}^{0} \frac{\sim}{\sim} = -i\pi a_{-1}$$

residue ... the a_{-1} coefficient of Laurent series.

Notice get $\frac{1}{2}$ the residue!

---

go back to our function

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_C f(z)\,dz$$

$$= \int_{-\infty}^{x_0-\delta} f(x)\,dx + \int_{C_{x_0}} f(z)\,dz + \int_{x_0+\delta}^{\infty} f(x)\,dx + \int_{\substack{\infty \\ \text{semi circle}}} f(z)\,dz$$

If C_{x_0} includes x_0 (we go below the x-axis) ccw

~~If x_0 excludes x_0 we go above~~

$$\oint f(z)\,dz = 2\pi i a_{-1} = \int_{-\infty}^{x_0-\delta} + i\pi a_{-1} + \int_{x_0+\delta}^{\infty} + \int_{\infty} f(z)\,dz$$

The one pole

If C_{x_0} excludes x_0 (we go above ...

$$\oint f(z)\,dz = \underset{\text{no poles}}{0} = \int_{-\infty}^{x_0-\delta} - i\pi a_{-1} + \int_{x_0+\delta}^{\infty} + \int_{\infty} f(z)\,dz$$

So either way we go

$$\boxed{\oint f(z)\,dz = \pi i\, a_{-1}}$$

Cauchy Principle Value

and ⭐ pole on path counts half! back to p 7 top

Now go back to integral

$$I_1 = \frac{1}{4n\pi}\int_{-\infty}^{\infty} g_{n+}(z)\, f(z)\,dz = \frac{1}{4n\pi}\int_{-\infty}^{\infty}\left(-\frac{2in}{z} + \dots\right) f(z)\,dz$$

$$= \frac{1}{4n\pi}\cdot\Big\{\overset{\text{(half a residue)}}{\pi i}(-2in)\,f(0) + 2\pi i \sum_p \mathrm{Res}\big[\underline{\underline{f(z_p)}}\, g_{n+}(z_p)\big]$$

these poles

$$= \frac{1}{2} f(0) + \frac{i}{2n}\sum_p \mathrm{Res}[\quad]$$

$\lim_{n\to\infty} \to 0$

$$\boxed{I_1 = \frac{1}{2} f(0)}$$

I_2 is the same except residue at $z=0$ is $+2in$ instead of $-2in$

~~so $I_2 = -\frac{1}{2} f(0)$~~ but because we integrate CW, we get another $-$ sign to cancel

recall ⭐

$$I_2 = \frac{1}{2} f(0)$$

So $$\int_{-\infty}^{\infty} \frac{1}{n\pi} \frac{\sin^2 nx}{x^2} f(x)\, dx = I_1 + I_2 = \underline{f(0)}$$

sifting condition

Implicit restrictions : $f(z) \to 0 \quad z \to \infty$

$f(z)$ only has simple poles

.... my take home is that $\frac{d}{dx}\delta$ is an expensive capability !

Derivative of the δ function

$$\int_{-\infty}^{\infty} \delta'(x)\, f(x)\, dx$$

WTF?

can't work with $\frac{d}{dx}\delta$! But we can use δ sequence

any differentiable δ sequence!

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} \underset{dv}{\phi_n'(x)}\, \underset{u}{f(x)}\, dx = \underbrace{\underset{v}{\phi_n(x)}\, \underset{u}{f(x)}\Big|_{-\infty}^{\infty}} - \lim_{n\to\infty}\int_{-\infty}^{\infty} \underset{v}{\phi_n(x)}\, \underset{du}{f'(x)\, dx}$$

for $f(x)$ bounded, $\rightarrow 0$ because $\phi_n(x) \rightarrow 0$ as $x \rightarrow \infty$ (agressively)

$$= -f'(0)$$

repeat process to n'th order

$$\boxed{\int_{-\infty}^{\infty} \delta^{(n)}(x)\, f(x)\, dx = (-1)^n f^{(n)}(0)}$$

An example An ideal dipole has moment $\vec{p} = p\hat{x}$ express charge density in terms of δ function

Electric potential

$$\Psi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d\vec{x}'$$

integral over charge dist.

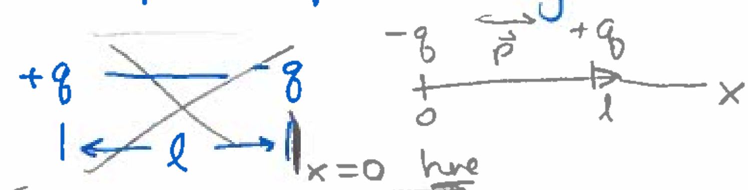
$dx\, dy\, dz$

Field point

charge locations

Find the electric potential due to a dipole at the origin ..

The usual approach for a dipole is paired charges



$$\rho(\vec{x}) = -q\,\delta(x)\,\delta(y)\,\delta(z) + q\,\delta(x-l)\,\delta(y)\,\delta(z)$$

$$= -q\,l\,\delta(y)\,\delta(z)\left[\frac{\delta(x) - \delta(x-l)}{l}\right]$$

we can make $q\,l = p$ the dipole moment

But we'll let $l \to 0$ and as $q \to \infty$ so that $q\,l = p$ remains finite ~~[that should be possible~~..

$$\rho(\vec{x}) = -p\,\delta(y)\,\delta(z) \lim_{l \to 0}\left[\frac{\delta(x) - \delta(x-l)}{l}\right]$$

ELFS?

$$\boxed{= -p\,\delta(y)\,\delta(z)\,\delta'(x)}$$ ⇒ my charge distribution

δ_n is not a function so this is a leap... could set as exercise and prove w/ δ sequence

$$\Psi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\,dV'$$

$$= \frac{1}{4\pi\epsilon_0}\iiint \frac{-p\,\delta'(x')\,\delta(y')\,\delta(z')}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\,dx'\,dy'\,dz'$$

$$= \frac{-p}{4\pi\epsilon_0}\int_{-\infty}^{\infty} \frac{\delta'(x')}{\sqrt{(x-x')^2 + y^2 + z^2}}\,dx'$$

Now, the δ' sifts for $-f'$ and we have $f(x) = \frac{1}{\sqrt{(x-x')^2+y^2+z^2}}$

$$= \frac{-p}{4\pi\epsilon_0} \cdot -f'(x')\Big|_{x'=0} = \frac{p}{4\pi\epsilon_0} \frac{(x-x')}{[(x-x')^2+y^2+z^2]^{3/2}}\Bigg|_{x'=0}$$

$$\psi(\vec{x}) = \frac{p}{4\pi\epsilon_0}\frac{x}{r^3}$$

end 29/1/19

What about the δ function of a function?

revisit top p 13

OMG

how does a δ function sift a function?

$$\int_{-\infty}^{\infty} \delta[g(x)]\, f(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n[g(x)]\, f(x)\, dx$$

change variable $u = g(x)$

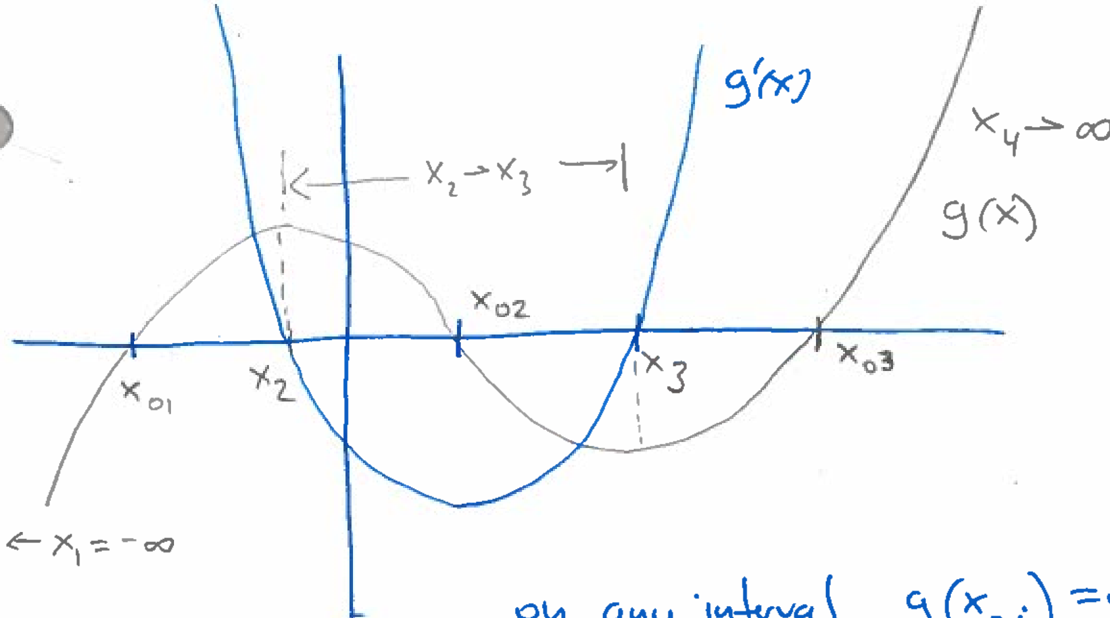
$du = g'(x)\,dx \longrightarrow dx = \frac{du}{g'(x)}$

Break integral up into segments when $g'(x) \neq 0$

this gives N segments where g' is either +ve or −ve and has 0 at the end points.

And it isolates only 1 at most 0's of $g(x)$ on any segment,

.... consider



on any interval $g(x_{oi}) = 0$

$x = x_{oi} \quad i = 1 \to N$

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So we carve up our function into intervals with 0's

$$\int_{-\infty}^{\infty} = \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} \phi_n[g(x)]\, f(x)\, dx = \sum_{i=1}^{N} \int_{g(x_i)}^{g(x_{i+1})} \phi_n(u)\, f[g^{-1}(u)] \frac{du}{g'(g^{-1}(u))}$$

$$\boxed{\begin{aligned} &\text{let } u = g(x_i) \\ &x_i = g^{-1}(u) \end{aligned}}$$

Yes
Σ

consider one of these integrals . . . $g'(x) > 0$ ~~x_i to x_{i+1}~~

x_i to x_{i+1}

because slope is everywhere > 0,

$$g(x_i) < g(x_{i+1})$$

as $n \to \infty$ $\phi_n \to \delta$ and fn equals 0 for any value outside some ϵ about $u=0$.

now just for i^{th} integral

$g^{-1}(u) = g^{-1}(g(x)) = x$

$u = g(x)$

$$\lim_{n\to\infty} \int_{g(x_i)}^{g(x_{i+1})} \phi_n(u)\, f[g^{-1}(u)] \frac{du}{g'(g^{-1}(u))}$$

It is understood that $\phi_n\, f[g^{-1}(u)]$ is 0 outside $x_i - x_{i+1}$ so $\int$ over ∞

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(u)\, f[g^{-1}(u)] \frac{du}{g'(g^{-1}(u))}$$

collapses to the $\underline{\underline{0}}$ between $x_{oi} - x_{i+1}$

$$= \frac{f[g^{-1}(0)]}{g'(g^{-1}(0))} = \frac{f(x_{oi})}{g'(x_{oi})}$$

+ve by construction

Now, for $g'(x) < 0$ $\quad x_i$ to x_{i+1}

$g(x_i)$ is $> g(x_{i+1})$

you get a flip sign in limits and get expression

$$= \frac{f(x_{oi})}{|g'(x_{oi})|}$$

$$\lim_{n\to\infty} \int_{g(x_i)}^{g(x_{i+1})} \phi_n(u)\, f(g^{-1}(u))\, \frac{du}{g'(g^{-1}(u))} = \lim_{n\to\infty} \int_{-\infty}^{-\infty}$$

$$= \lim_{n\to\infty} - \int_{-\infty}^{\infty} \phi_n(u)\, f\left[g^{-1}(u)\right] \frac{du}{g'(g^{-1}(u))}$$

$$= \frac{-f(g^{-1}(0))}{g'(g^{-1}(0))} = \frac{\cancel{+}\, f(x_{0i})}{|g'(x_{0i})|}$$

< 0 ! by construction

the leading − sign absorbs sign of denominator

$$\int_{-\infty}^{\infty} \delta(g(x))\, f(x)\, dx = \sum_{i=1}^{N} \frac{f(x_{0i})}{|g'(x_{0i})|}$$

sum over the 0's of $g(x)$

Can also express as

$$\delta(g(x)) = \sum_{i=1}^{N} \frac{\delta(x - x_{0i})}{|g'(x_{0i})|}$$

~~There has to be an example~~

⇒ need a finite number of 0's

- also fails for repeated roots

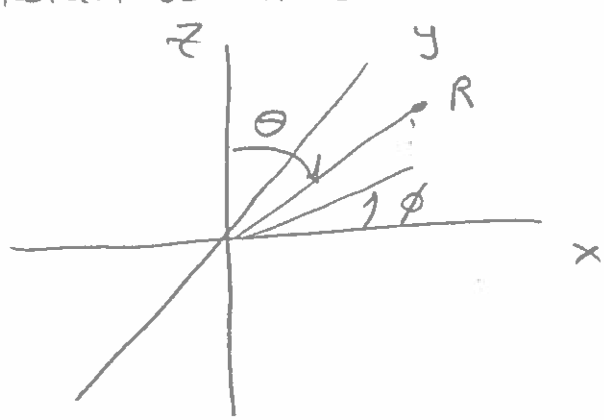
$\subset$ $g'(x_{0i}) \to 0$! ack!

bad

A sheet of charge lies in the $z=0$ plane with surface charge density σ_0. Express the VOLUME charge density in spherical coordinates ....

oh shit

easy in cartesian but now what?



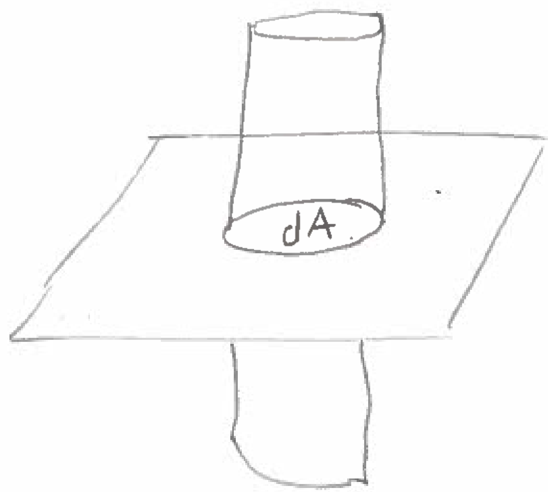
The constraint of charge at $z=0$ plane invites the δ function.

$$\rho(\vec{x}) = k\,\delta(z)$$

$\hookrightarrow$ k could be fn of x, y, z

we have to find $\underline{\underline{k}}$

this is easy in cartesian/cylindrical ... consider a unit cross section cylinder going from $-\infty$ to ∞



$$dq = \int_{\text{cylinder}} \rho(\vec{x})\, dV = \int_{-\infty}^{\infty} k\,\delta(z)\, dz\, dA = k\, dA = \sigma_0\, dA$$

$$\underline{\underline{k = \sigma_0}}$$

you could have guessed that

$$\boxed{\rho(\vec{x}) = \sigma_0\, \delta(z)}$$

now spherical ...

$$\rho(\vec{x}) = \sigma_0 \, \delta(r\cos\theta) \qquad 0 < \theta < \pi$$

okay... the 0's will occur at $\theta = \pi/2$ on this interval but what of $\underline{\underline{r}}$?

Charge exist $\forall$ values of r so can't have it as a δ function. It is essentially a parameter!

$$\delta(ax) = \frac{\delta(x)}{a}$$

$$\rho(\vec{x}) = \frac{\sigma_0}{r}\, \delta(\cos\theta)$$

lordy be we have a δ function of a function and that function has 1 0 that occurs at $\theta = \frac{\pi}{2}$

$$\rho(\vec{x}) = \sigma_0 \frac{\delta(\theta - \pi/2)}{r \, |-\sin\theta|_{\theta=\frac{\pi}{2}}} = \boxed{\frac{\sigma_0}{r}\, \delta\left(\theta - \frac{\pi}{2}\right)}$$

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start here 6/2/19

$$\rightarrow \quad \rho(\vec{x}) = \frac{\sigma_0}{r}\, \delta\left(\theta - \frac{\pi}{2}\right)$$

key points... δ localises (isolates) the $\theta = \frac{\pi}{2}$ plane
r is like a constant because it doesn't isolate charge.

It is good to test the result to see if it acts as expected. Take a spherical shell of thickness dr, the shell intersects the charge sheet in a plane and you get

$$dq = \boxed{dr \cdot 2\pi r \cdot \sigma_0}$$

... does our polar charge distribution agree?

$$dq = \int_{shell} \rho(\vec{r})\, dv = \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \overbrace{\frac{\sigma_0}{r}\,\delta\!\left(\theta-\frac{\pi}{2}\right)}^{\rho(\vec{r})} \underbrace{r^2 \sin\theta\, d\theta\, dr}_{dV_{polar}}$$

$$= 2\pi \cdot \frac{1}{r}\,\sigma_0\, r^2 \sin\frac{\pi}{2}\, dr = \boxed{2\pi\sigma_0\, r\, dr}$$

Ok

We will return to this mass / charge distributions with ~~for~~ δ constraints again. But

The general approach is

1. find all coordinates that take on single value $\Rightarrow$ assign δ's
2. assign an unknown density function $C(x, y, z \ldots)$
3. solve charge in a known geometry to establish C

$\int$ of the δ function

↳ sort of defined but use block function:

$$\int_{-\infty}^{x} \phi_n(u)\,du = 0 \qquad x < -\frac{1}{n} \qquad \text{yes!}$$

$x > \frac{1}{n}$

$$\int_{-\infty}^{x} \phi_n(u)\,du = \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{n}{2}\,du = \frac{un}{2}\Bigg|_{-\frac{1}{n}}^{\frac{1}{n}} = 1$$

$$\int_{-\infty}^{x} \phi_n(u)\,du = \begin{cases} 0 & x < -\frac{1}{n} \\ 1 & x > \frac{1}{n} \end{cases}$$

This can be squeezed arbitrarily

this is a distribution!

$$\lim_{n\to\infty} \int = \int_{-\infty}^{x} \delta(u)\,du = \begin{cases} 0 & x \leq 0 \\ 1 & x > 1 \end{cases} \equiv \Theta(x) = S(x)$$

not defined at $0 = x$

step fn

We know the step function:

$$\int_{-\infty}^{\infty} f(x)\,\Theta(x)\,dx = \int_{0}^{\infty} f(x)\,dx$$

↳ and it kind of follows that $\underline{\frac{d\Theta}{dx} = \delta(x)}$

Fourier series of δ function

(Consider block model)

assume Fourier form

$$\phi_n(x) = \sum_{m=-\infty}^{\infty} c_m \, e^{i m \pi x / L}$$

delta sequence

(valid on interval $-L < x < L$

$e^{-i\cdots}$ gives c_m

$$c_m = \frac{1}{2L} \int_{-L}^{L} \phi_n(x) \, e^{-i m \pi x / L} \, dx$$

$$= \frac{1}{2L} \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{n}{2} \, e^{-i m \pi x / L} \, dx = \frac{1}{2L} \frac{n}{2} \frac{-L}{i m \pi} \, e^{-i m \pi x / L} \Big|_{-\frac{1}{n}}^{\frac{1}{n}}$$

$$c_m = \frac{-n}{4 \pi m} \left[\frac{e^{-i m \pi / nL} - e^{+ i m \pi / nL}}{i} \right]$$

$$c_m = \frac{-n}{2 \pi m} \sin \frac{-m\pi}{nL} = \frac{n}{2 \pi m} \sin \frac{m \pi}{nL}$$

important else ϕ_n when is the n

$$\lim_{n \to \infty} \frac{n}{2 \pi m} \cdot \frac{m \pi}{n L} = \boxed{\frac{1}{2L}}$$

And so

$$\boxed{\delta(x) = \frac{1}{2L} \sum_{m=-\infty}^{\infty} e^{i m \pi x / L}}$$

has got differentiation properties!

coefficients are constant!

– notice that the coefficients do not decrease as m increases → hmm doesn't really converge … might return to that (this sort of thing will come up again)

… Does it sift?

$$\int_{-L}^{L} \frac{1}{2L} \sum_{m=-\infty}^{\infty} e^{im\pi x/L} f(x)\, dx$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{\frac{1}{2L} \int_{-L}^{L} e^{im\pi x/L} f(x)\, dx}$$

~~But~~ this will ~~only~~ give the a_{-m} coefficients of the Fourier series for $f(x)$?

ie) $f(x) = \sum_{p=-\infty}^{\infty} a_p e^{ip\pi x/L}$ → $a_p = \frac{1}{2L} \int_{-L}^{L} e^{-ip\pi x/L} \cdot f(x)\, dx$

So the sum collapses to

omit

omit or add later

$$= \sum_{m=-\infty}^{\infty} a_{-m} \cdot 1 = \sum_{m=-\infty}^{\infty} a_m \cdot \boxed{1} = f(0)$$

$a_{-m} = a_m$ f real

$$e^{ip\pi x/L}\Big|_{x=0} = 1$$

The sum of the coefficients is the same as Σ coefficient $\cdot 1$ at each station which corresponds to $\Sigma a_n \cdot 1$ → gives $f(0)$!

end here 6/2/19

Example

An initially stationary string stretched to some length L is hit with an impulsive force at a point $L/3$ -- find the subsequent motion of the string....

- recall wave speed on string $v = \sqrt{T/\mu}$
- what is the impulse applied to the string as a function of position

$$= \underbrace{I_0}_{\text{magnitude}} \, \delta\left(x - \frac{L}{3}\right)$$

location

impulse / unit length string

$I = \Delta p$ the change in momentum of the string element

$$d(mv) = \mu \, dx \cdot v_{\text{initial}}$$

describe string position as $y(x,t)$ and initial velocity $\frac{\partial}{\partial t} y(x, t = t_0)$

$$m \cdot v = \mu \, dx \cdot \frac{\partial}{\partial t} y(x,t)\Big|_{t=t_0} = \underbrace{I_0 \, \delta\left(x - \frac{L}{3}\right)}_{\text{Impulse per unit length}} \underbrace{dx}_{dx}$$

$$\Rightarrow \frac{\partial}{\partial t} y(x,t)\Big|_{t=0^+} = \frac{I_0}{\mu} \, \delta\left(x - \frac{L}{3}\right)$$

avoid $t=0$

We assume the solution has the form:

$$y(x,t) = \sum_{n=1}^{\infty} a_n \underbrace{\sin \frac{n\pi x}{L}}_{\text{0 at boundaries}} \underbrace{\sin \frac{n\pi v t}{L}}_{\text{SHM}}$$

[Why? → wave consistent → Fourier fits anything (ish)

and meets $y(x,0)=0$ constraint

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \frac{n\pi v}{L} \cos \frac{n\pi v t}{L}$$

and at $t=0$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} a_n \frac{n\pi v}{L} \sin \frac{n\pi x}{L}$$

And this must agree with

$$\frac{I_0}{\mu} \delta\left(x - \frac{L}{3}\right) = \sum_{n=1}^{\infty} \underbrace{\left(a_n \frac{n\pi v}{L}\right)}_{\text{all of this will be the coefficient}} \sin \frac{n\pi x}{L}$$

Use orthogonality to find $a_n \frac{n\pi v}{L}$ $\quad \int_0^L \sin \frac{n\pi x}{L} \, dx$

$$a_n \frac{n\pi v}{L} = \frac{2}{L} \int_0^L \frac{I_0}{\mu} \delta\left(x - \frac{L}{3}\right) \sin \frac{n\pi x}{L} \, dx$$

$$= \frac{2}{L} \frac{I_0}{\mu} \sin \frac{n\pi 3}{L} \quad \rightarrow \quad a_n = \frac{2 I_0}{n\pi v \mu} \sin \frac{n\pi 3}{L}$$

TOP P 3.3

$$y(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{n\pi v} \frac{I_0}{\mu} \sin\frac{n\pi}{3} \underbrace{\sin\frac{n\pi x}{L}}_{\text{spatial}} \underbrace{\sin\frac{n\pi v t}{L}}_{\text{temporal}} \right]$$

$(kx - \omega t)$

look at graphs

why i̲s̲n̲'̲t̲ $y(x,t)\big|_{t=0} = 0 \quad \forall \; x \neq \frac{x}{L} = \frac{1}{3}$

→ finite wave speed! Important

→ amplitude of $N=20$ result?

→

```
clear

x = 0:.00001:1;
L = 1;

N = 10; vtoverl = .5;
yxt = zeros(size(x));
for n = 1:N
    yxt = yxt + 2/n/pi*( sin(n*pi/3)) .* sin(n*pi*x/L) .* sin(n*pi*vtoverl);
end

figure(1)
clf
plot(x,yxt)
hold on


N = 200;
yxt = zeros(size(x));
for n = 1:N
        yxt = yxt + 2/n/pi*( sin(n*pi/3)) .* sin(n*pi*x/L) .* sin(n*pi*vtoverl);
end

figure(1)
plot(x,yxt,'r')
legend('N = 20','N = 200')
xlabel('x/L')
ylabel('I / (v \mu)')
ylabel('y / (I / (v \mu))')
title('v t / L = 0.1')


N = 20; vtoverl = .01;
yxt = zeros(size(x));
for n = 1:N
    yxt = yxt + 2/n/pi*( sin(n*pi/3)) .* sin(n*pi*x/L) .* sin(n*pi*vtoverl);
end

figure(2)
clf
plot(x,yxt)
hold on


N = 200;
yxt = zeros(size(x));
for n = 1:N
    yxt = yxt + 2/n/pi*( sin(n*pi/3)) .* sin(n*pi*x/L) .* sin(n*pi*vtoverl);
end
```

```
figure(2)
plot(x,yxt,'r')
legend('N = 20','N = 200')
title('v t / L = 0.01')
xlabel('x/L')
ylabel('y / (I / (v \mu))')
```

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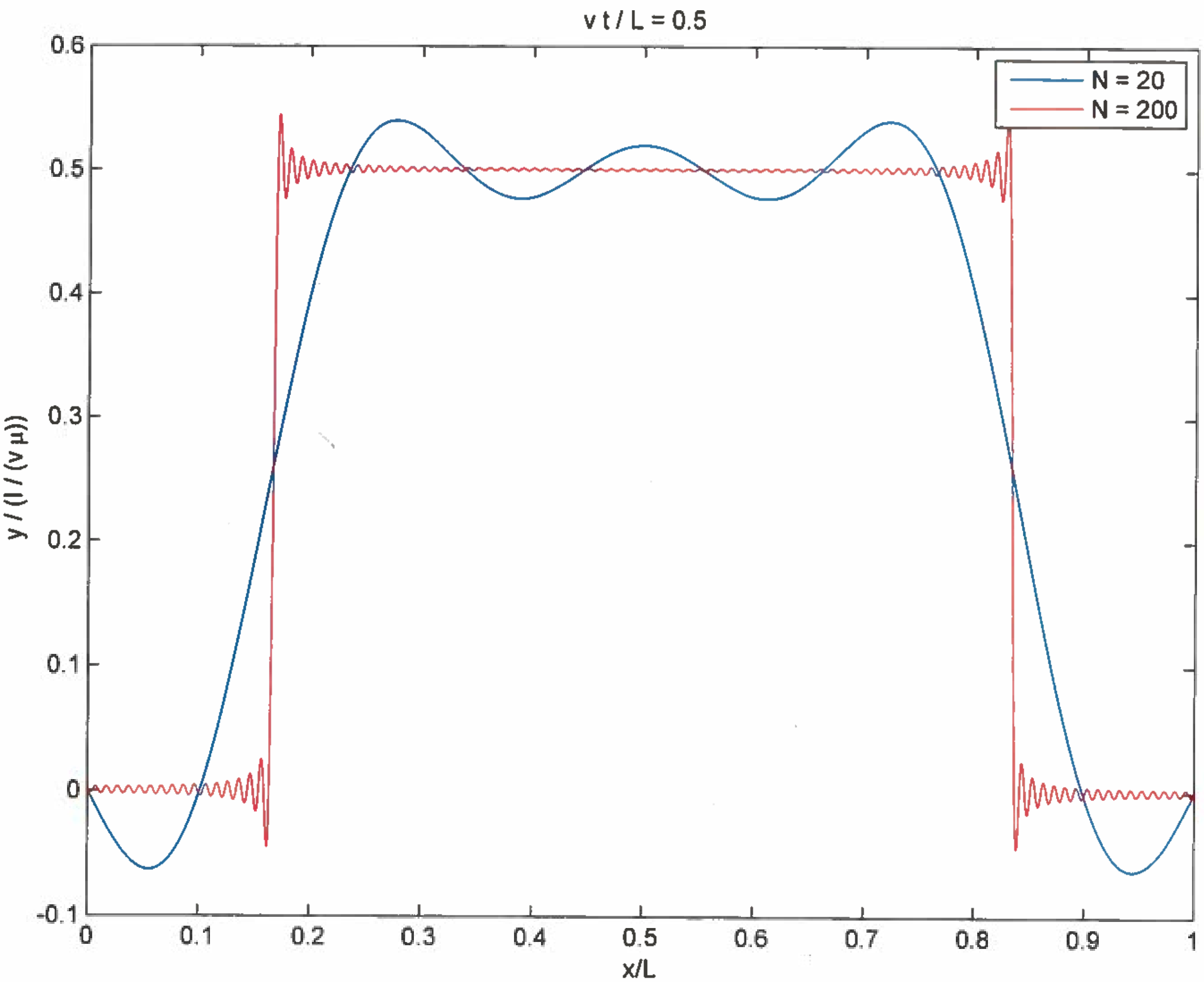
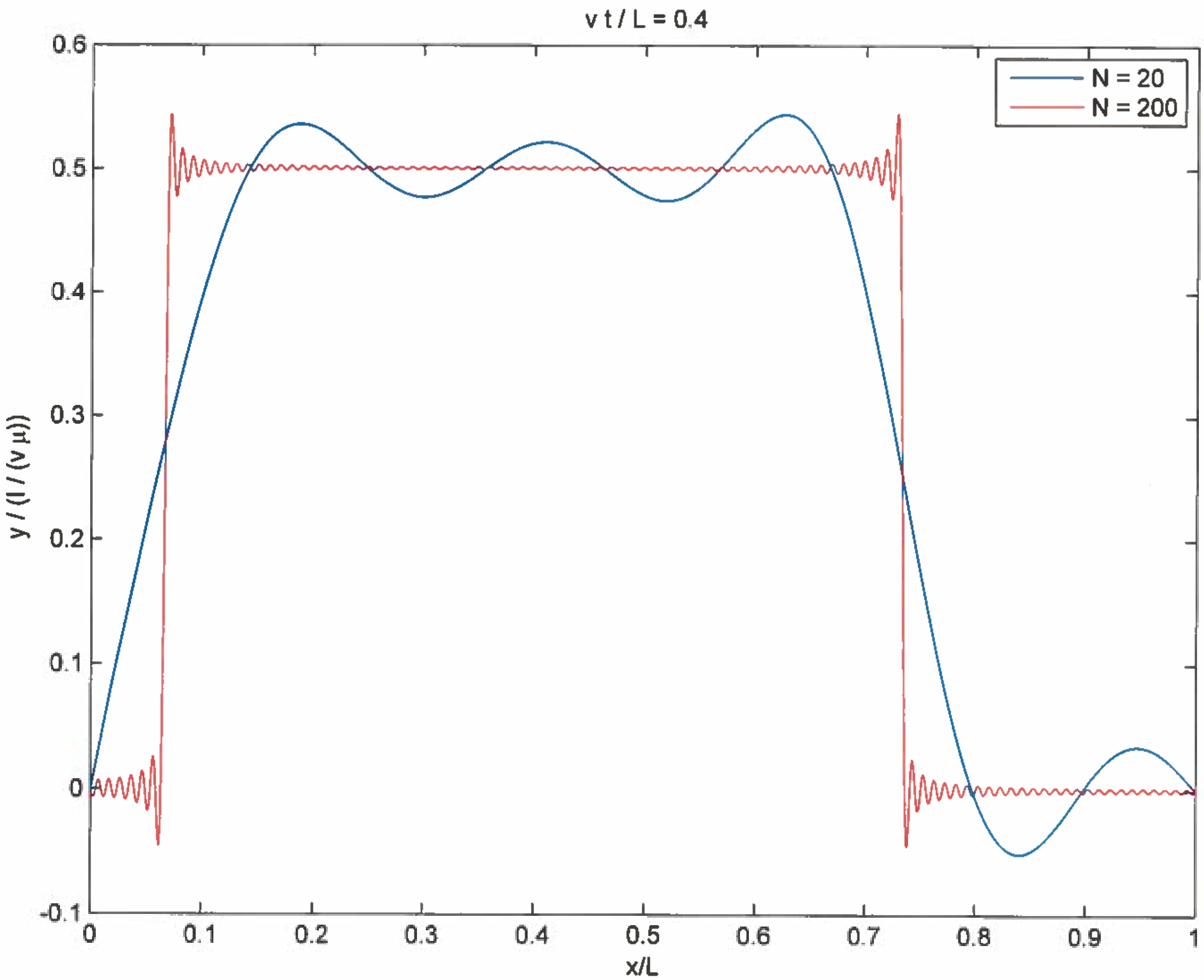
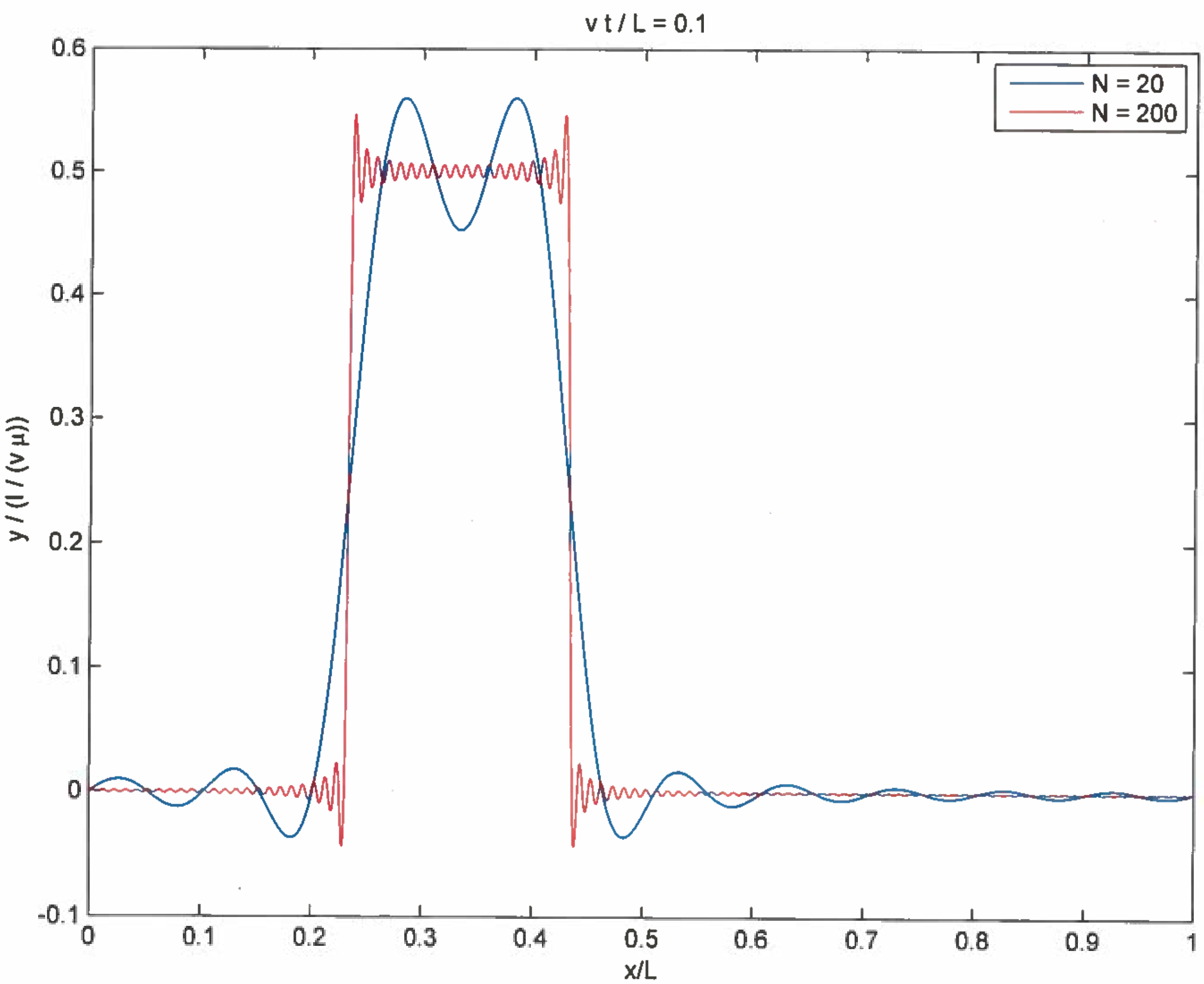
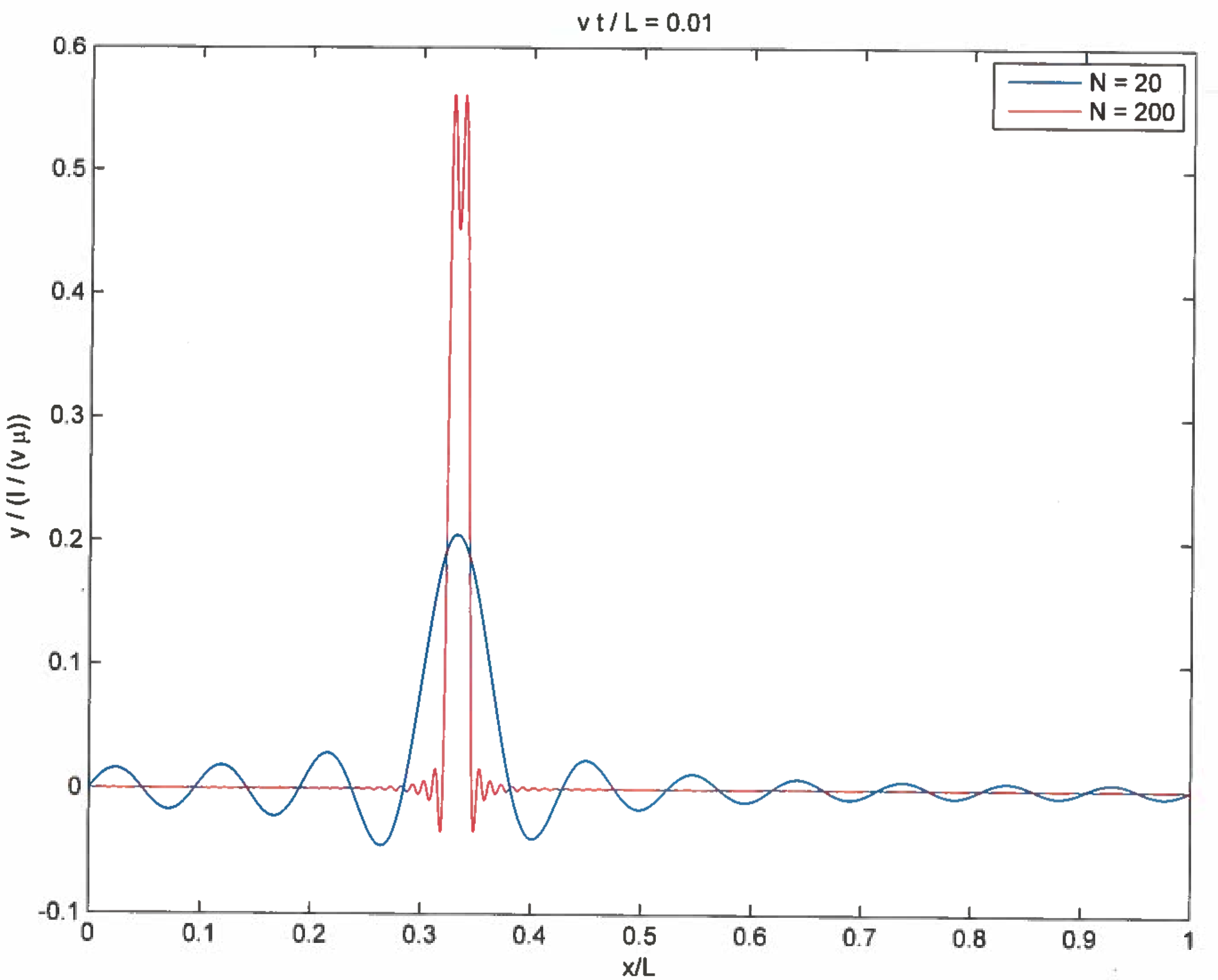
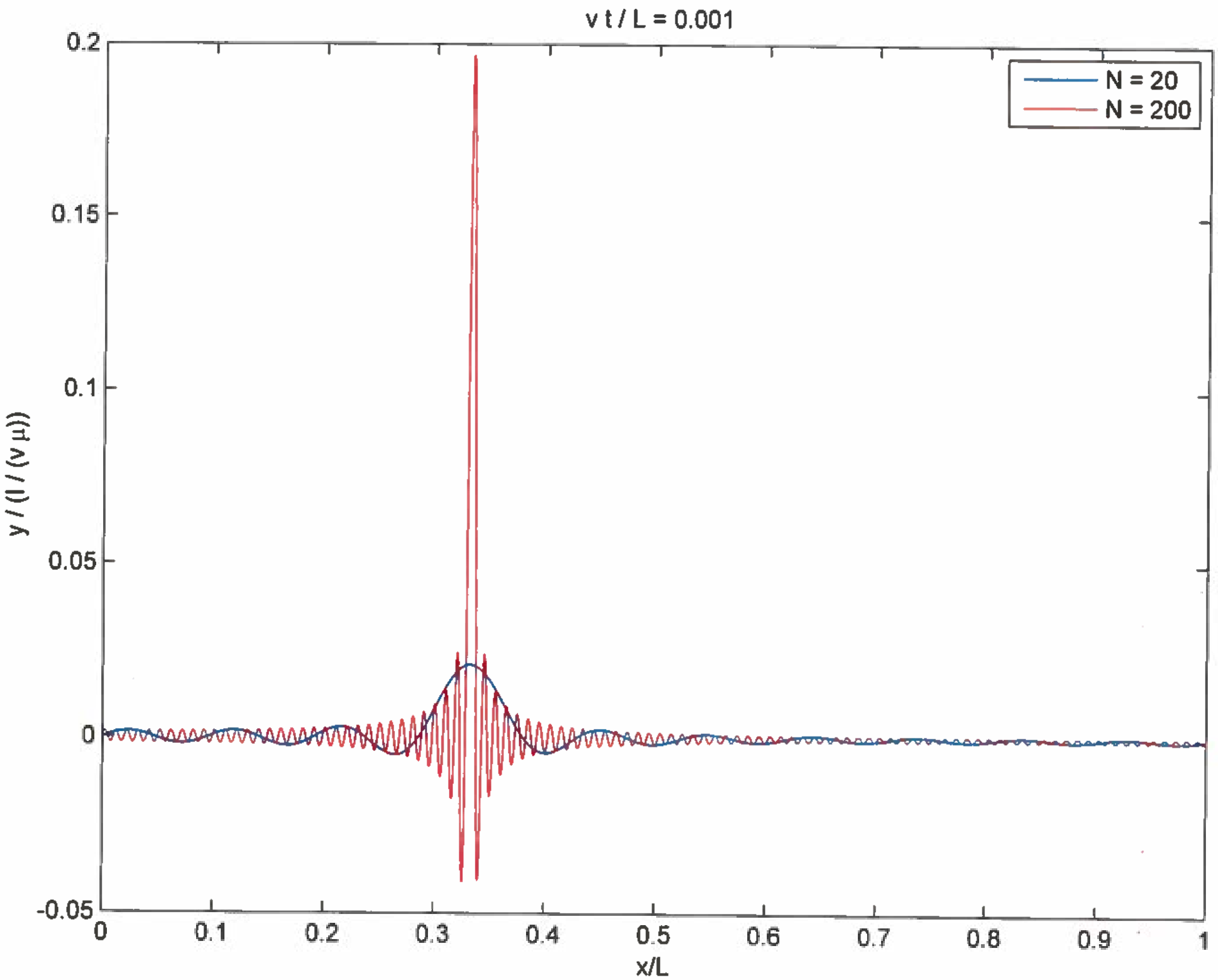

v t / L = 0.5
N = 20
N = 200
y / (I / (v μ))
x/L
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



v t / L = 0.4
N = 20
N = 200
y / (I / (v μ))
x/L
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


v t / L = 0.1
N = 20
N = 200
y / (I / (v μ))
x/L
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


v t / L = 0.01
N = 20
N = 200
y / (I / (v μ))
x/L
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


v t / L = 0.001
N = 20
N = 200
y / (I / (v μ))
x/L
0.2
0.15
0.1
0.05
0
-0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1