Steve PAL ask: ①

ended with

$$Q(s) = \frac{\varepsilon}{L}\frac{1}{s}\frac{1}{\left(\frac{1}{LC}-\frac{1}{4}\frac{R^2}{L^2}\right)^{1/2}}\frac{\left(\frac{1}{LC}-\frac{1}{4}\frac{R^2}{L^2}\right)^{1/2}}{\left(s+\frac{R}{2L}\right)^2+\left(\frac{1}{LC}-\frac{1}{4}\frac{R^2}{L^2}\right)}$$

big simplification let $\left(\frac{1}{LC}-\frac{1}{4}\frac{R^2}{L^2}\right) = \omega^2$

$$Q = \frac{\varepsilon}{L}\cdot\frac{1}{s}\cdot\frac{1}{\omega}\cdot\frac{\omega}{\left(s+\frac{R}{2L}\right)^2+\omega^2}$$

$\to \sin\omega t$

constant

stand by constant

translation

$e^{-R/2L\,t}$

So what about the stray s

recall $i = \frac{dq}{dt} \Rightarrow \mathcal{L}(i) = \mathcal{L}\left[\frac{dq}{dt}\right] = sQ - q_0$

s accounted for.

$$I = sQ = \frac{\varepsilon}{L\omega}\cdot\frac{\omega}{\left(s+\frac{R}{2L}\right)^2+\omega^2}$$

$$\boxed{i(t) = \frac{\varepsilon}{L\omega}e^{-R/2L\,t}\sin\omega t}$$

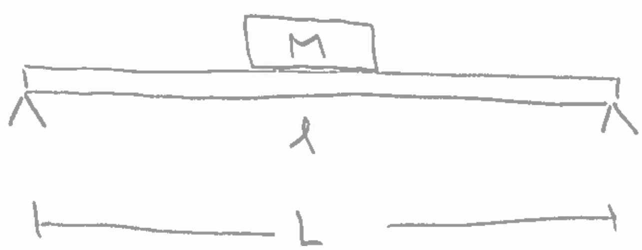
There are some standard circuit analysis short cuts you can make:

$$\frac{R}{2L} = \alpha \quad \text{damping characteristic of RL circuit}$$

$$\sqrt{\frac{1}{LC}} = \omega_0 \quad \text{natural frequency of LC circuit}$$

→ I don't like making substitutions until you know what they mean physically. ... Sure look for substitutions but be prepared to re pack the evaluation later.

Beam across gap



How is beam deflected?

beam deflection regulated by

$$\frac{d^4 y}{dx^4} = \frac{1}{E\ I}\ q(x)$$

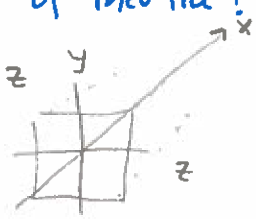
Young's modulus (spring constant)

moment of inertia!

supported force as f(x)

$\iint y^2 \, dy\, dz$

slice beam



How do I make I big?

$I = \frac{z\, y^3}{12}$ rectangle

what about the 4'th order derivative? ③

- beam bends as a result of torque

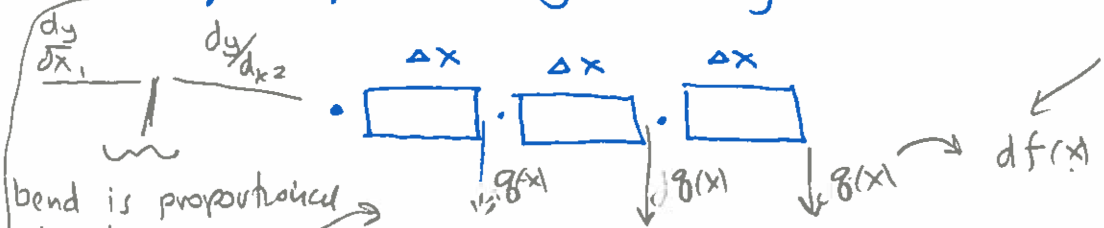
$$\frac{d^2y}{dx^2} = -\frac{1}{EI}\, m(x)$$

a CHANGE in slope indicates some additional torque

net ccw torque to right of a point x

y increased · downward

Torque is provided by shearing forces



$m(x) = \int q(x) \cdot x\, dx$

$\Delta x\, q(x) + 2\Delta x\, q(x) + 3\Delta x\, q(x) = 6\Delta x\, q(x)$

- $$\frac{dm}{dx} = t(x)$$ ← all downward force to right of x

change in slope $$\frac{d^3y}{dx^3} = -\frac{1}{EI}\, t(x)$$

Shear force results from load per unit length

$$\frac{dt}{dx} = -\underbrace{q(x)}_{\text{a local property}} \rightarrow t = \int_x^L q(x')\,dx'$$

load per unit length

- change in change in slope $$\boxed{\frac{d^4y}{dx^4} = +\frac{1}{EI}\, q(x)}$$

Okay 4'th order B equn $\Rightarrow$ 4 boundary conditions

As defined $y(0) = y(L) = 0$ height at ends (2 b.c.'s)

Also, at $x=0$ there can be no torque (there is no rod to support it)

$$\left.\frac{\partial^2 y}{\partial x^2}\right|_{0,L} = 0 \qquad (2 \text{ more b.c.'s})$$

We have the load in the middle of the beam:

$$q(x) = \begin{cases} Mg/\ell & (L-\ell)/2 < x < (L+\ell)/2 \\ 0 & \text{else} \end{cases}$$

Transform our equns with $\mathcal{L}$ $\quad \frac{d^4 y}{dx^4} = \frac{1}{EI} q(x)$

$$s^4 Y - s^3 y(0) - s^2 y'(0) - s\, y''(0) - y'''(0) = \frac{1}{EI} Q(s)$$

$$Q(s) = \int_0^\infty q(x) e^{-sx} dx = \int_0^{(L-\ell)/2} + \int_{\frac{L-\ell}{2}}^{\frac{L+\ell}{2}} + \int_{\frac{L+\ell}{2}}^{L} \frac{Mg}{\ell} e^{-sx} dx$$

$$= \frac{-Mg}{s\ell}\left[e^{-\frac{(L+\ell)}{2}s} - e^{-\frac{(L-\ell)}{2}s} \right]$$

$$= \frac{+Mg}{s\ell} e^{-Ls/2} \cdot 2 \sinh \ell s/2$$

So put this all into equ

$$s^4 Y - s^2 y'(0) - y'''(0) = \frac{+Mg}{EIs\ell} e^{-Ls/2} \sinh \ell s/2$$

solve for Y

$$Y(s) = \frac{Mg}{EI s^5 \ell} \underbrace{e^{-\frac{Ls}{2}} \cdot 2\sinh \ell s/2}_{\text{translation?}} + \frac{y'(0)}{s^2} + \frac{y'''(0)}{s^4}$$

$\frac{Mg}{EI s^5 \ell} \downarrow$ powers of x $\frac{x^4}{4!}$

end here 14/1/19

unpack $e^{-Ls/2} \cdot \sinh$ again...

$$-\left[e^{-(L+\ell)s/2} - e^{-(L-\ell)s/2} \right]$$

translation by $\frac{L+\ell}{2}$ $\frac{L-\ell}{2}$

e^{-st_0} gives $f(t-t_0)$ with $S(t-t_0)$ to control $t < t_0$

$$y(x) = \frac{Mg}{EI\ell}\frac{1}{4!}\left[-\left(x - \frac{L+\ell}{2}\right)^4 S\left(x - \frac{L+\ell}{2}\right) + \left(x - \frac{L-\ell}{2}\right)^4 S\left(x - \frac{L-\ell}{2}\right) \right] + y'(0)x + y'''(0)\frac{x^3}{3!}$$

Evaluate at $y(L) = 0$ knowing $y''(L) = 0$

$y(L) = 0 \Rightarrow y'''(0) = \dfrac{-Mg}{2EI}$ requires

$y''(L) = 0 \Rightarrow y'(0) = \dfrac{Mg}{EI} \dfrac{3L^2 - \ell^2}{48}$

* L is the chord length not the rest length

* load is distributed evenly -- not always the case

* our beam has no mass!

Some more properties

Derivative of transform

$$\frac{dF}{ds} = \frac{d}{ds}\int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty f(t)\, \frac{d}{ds} e^{-st}\, dt$$

$$= -\int_0^\infty t \cdot f(t)\, e^{-st}\, dt = -\mathcal{L}\left[t\, f(t)\right]$$

$$\boxed{-\mathcal{L}\left[t\, f(t)\right] = \frac{dF}{ds}}$$

and by extension

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n F}{ds^n}$$

So for example:

$$\mathcal{L}\left[t \sin\omega t\right] = -\frac{d}{ds}\left[\frac{\omega}{s^2+\omega^2}\right]$$

$$= \frac{2 s\omega}{(s^2+\omega^2)^2}$$

Integral of transform

$$\int_s^\infty F(\sigma)\,d\sigma = \int_{\sigma=s}^{\infty}\int_{t=0}^{\infty} f(t)\,e^{-\sigma t}\,dt\,d\sigma$$

$\int$ over σ first

obviously this limit must $\to 0$

$$= \int_0^\infty -f(t)\,\frac{e^{-\sigma t}}{t}\Big|_s^{\infty}\,dt$$

$$= \int_0^\infty \frac{f(t)}{t}\,e^{-st}\,dt$$

$$\boxed{\int_s^\infty F(\sigma)\,d\sigma = \mathcal{L}\left[\frac{f(t)}{t}\right]}$$

Example application of $\int$

Suppose we ~~want~~ need to transform

$$f(t) = \frac{1-\cos t}{t}$$

well, just look at $\mathcal{L}[1-\cos t] \Rightarrow \frac{1}{s} - \frac{s}{s^2+\omega^2}$

now $$\mathcal{L}\left[\frac{1-\cos t}{t}\right] = \int_s^\infty \left[\frac{1}{\sigma} - \frac{\sigma}{\sigma^2+\omega^2}\right] d\sigma$$

$$= -\ln s + \frac{1}{2}\ln(s^2+\omega^2)$$

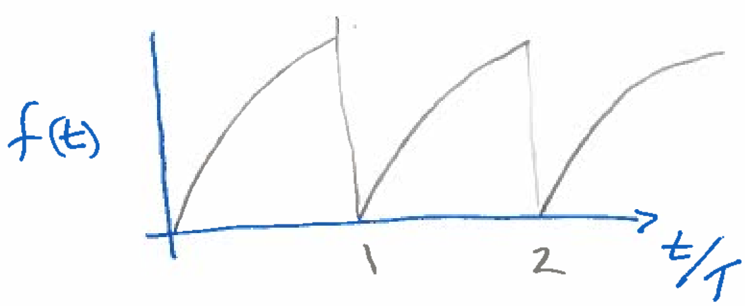
what about $s \to \infty$?

$$\lim_{s\to\infty} \ln s - \frac{1}{2}\ln(s^2+\omega^2) = \lim_{s\to\infty} \ln s - \frac{1}{2}\ln s^2$$

$$= \lim_{s\to\infty} \ln s - \ln s \Rightarrow 0!$$

OMG

Periodic functions



so that

$$f(t) = \begin{cases} g(t) & 0 \le t < T \\ g(t-T) & T \le t < 2T \\ g(t-2T) & 2T \le t < 3T \end{cases}$$

hmmm ... equivalent to a sum of functions offset by offsets

$$f(t) = g(t) + S(t-T)\, g(t-T) + S(t-2T)\, g(t-2T) + \cdots$$

where $g(t) \rightarrow$ 

let $$G(s) = \int_0^\infty g(t)\, e^{-st}\, dt = \int_0^T g(t)\, e^{-st}\, dt$$

for the second cycle:

$$\mathcal{L}\left[S(t-T)\, g(t-T)\right] = e^{-sT}\, G(s)$$ shifting property

$$F(s) = \left[1 + e^{-sT} + e^{-2sT} + \cdots\right] G(s)$$

geometric series

$$F(s) = \left[\frac{1}{1 - e^{-sT}}\right] G(s)$$

and, any transform with a denominator of this form is periodic

Convolution

why: i) it's conceptually interesting
ii) It will allow us to see how integrations effect the transform.

→ Convolutions arise when the present state of a system (function/solution) depends on previous states. This is often the case with space/time evolving systems.

end 16/1/19

Reconsider RLC circuit for which

$$L\frac{di}{dt} + Ri + \frac{q}{c} = \mathcal{E}(t)$$

$$i = \frac{dq}{dt} \longrightarrow I = sQ - q(0)$$

let $i(0) = 0$ & $q(0) = 0$

$$s^2QL + sRQ + \frac{Q}{C} = E(s)$$

$$Q = \frac{E(s)}{s^2L + sR + 1/C} = E(s)\,R(s)$$

what you're driving it with

system response

where $R(s) \equiv \dfrac{1}{s^2L + sR + 1/C} = \dfrac{1}{L}\dfrac{1}{(s+\alpha)^2 + \omega^2} = \mathcal{L}[r(t)]$

$$\alpha = \frac{R}{2L} \qquad \omega = \sqrt{\omega_0^2 - \alpha^2} \qquad \omega_0 = \sqrt{1/LC}$$

Q can be thought of as the product of two transforms

product in s space $\longleftrightarrow$ convolution in t space

$$\mathcal{L}^{-1}[E \cdot R] = \mathcal{E} * r = \int_0^t r(\tau)\, \mathcal{E}(t-\tau)\, d\tau$$

notice -- $\mathcal{E} * r = r * \mathcal{E}$

a convolution integrates ~~everything~~ things that has happened ($\mathcal{E}(t-\tau)$) up until Now ie $0 \to t$

for charge on capacitor problem

$$q(t) = \int_0^t \mathcal{E}(\tau)\, r(t-\tau)\, d\tau$$

when we analysed this before we effectively found the transform of R(s) corresponding to

$$r(t) = \frac{1}{L\omega} e^{-\alpha t} \sin \omega t \quad \text{(damped oscilator)}$$

but that was for a very simple driving fn $\mathcal{E}$

what about something more complicated?

Try: $\mathcal{E}(\tau) = \mathcal{E}_0 \cos \Omega \tau$ an oscilator

$$\mathcal{L}[\mathcal{E}] = \mathcal{E}_0 \frac{s}{s^2 + \Omega^2} \Rightarrow Q(s) = ER = \frac{\mathcal{E}_0}{L} \frac{s}{s^2+\Omega^2} \cdot \frac{1}{(s+\alpha)^2 + \omega^2}$$

would be interesting to try it!

Convolution

$$q(t) = \int_0^t \frac{\mathcal{E}_0}{L\omega} \cos \Omega t \sin[\omega(t-\tau)] e^{-\alpha(t-\tau)} d\tau$$

$$= \frac{\mathcal{E}_0}{L\omega} e^{-\alpha t} \int_0^t \cos \Omega\tau \sin \omega(t-\tau) e^{\alpha\tau} d\tau$$

– go to old p 21

⋮

– Take home message:

For large t, circuit has response frequency Ω determined by forcing function and the circuit characteristics determine the phase. (ω, α, Ω)

For small t $(t \ll 1/\alpha)$ the $e^{-\alpha t}$ dominates and you get damped oscillator.

Okay ... just to something different:

The transform of an integral

Consider a function $g(t) \equiv 1$ $G(s) = 1/s$

$$\mathcal{L}\left[\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right] = F(s) \cdot G(s)$$

So using convolution theorem and our choice of $g(t)$

$$\boxed{\mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = F(s) \cdot \frac{1}{s}}$$

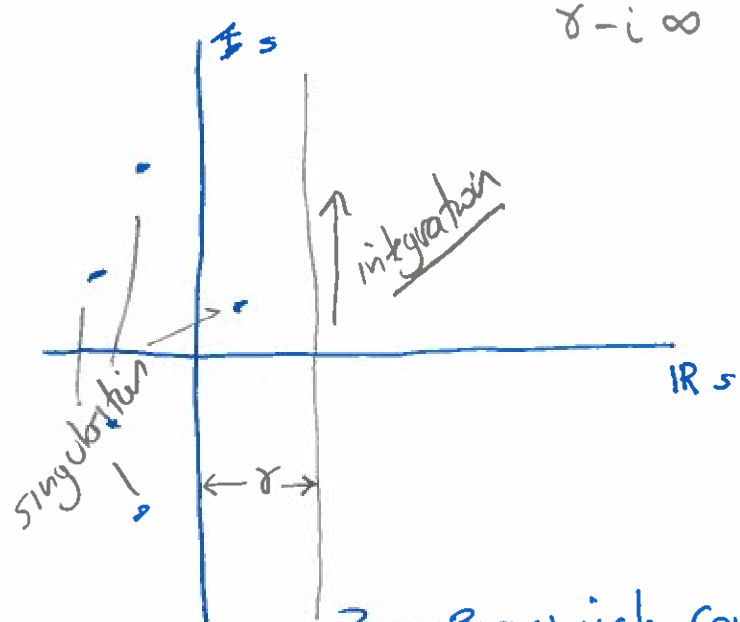
"primitive function theorem"

The general inversion procedure

– what to do if you don't know the solution by inspection

Mellin Inversion Integral

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s)\, e^{st}\, ds \qquad *$$



→ Bromwich contour

⟶ all singularities in $F(s)$ must be to left of Bromwich ←

Proof: take the transform of $*$ and recover $F(s)$!

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} \cdot \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(\sigma)\, e^{\sigma t}\, dt\, d\sigma$$

$t \qquad \sigma$

reverse order ... integration

$$= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(\sigma) \int_{0}^{\infty} e^{-st+\sigma t} \, dt \, d\sigma$$

$\sigma \qquad t$

$$= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(\sigma) \left. \frac{e^{-(s-\sigma)t}}{\sigma - s} \right|_{0}^{\infty} d\sigma$$

For all pts ... to right of Bromwich

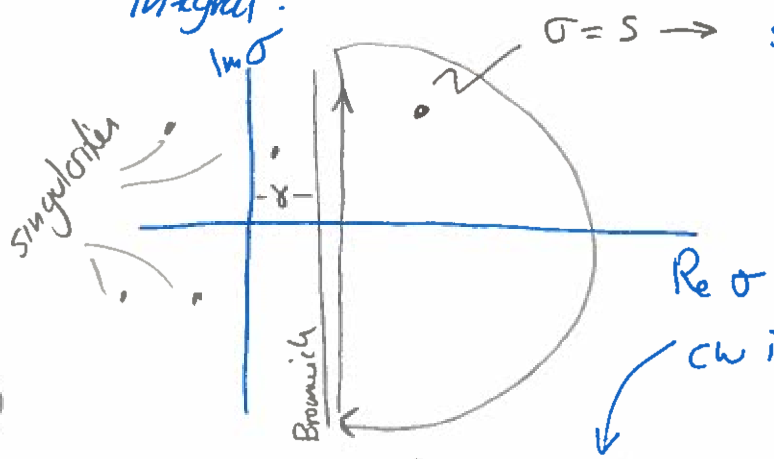
$$Re(s) > Re(\sigma) = \gamma \quad e^{-(s-\sigma)t} \longrightarrow 0 \text{ as } t \to \infty$$

and

$$= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{F(\sigma)}{s - \sigma} \, d\sigma$$

– sign from lower limit

$F(\sigma)$ is analytic everywhere to right of γ (we arranged this by excluding all singularities) so we can use a closed integral:



$\sigma = s \rightarrow$ single pole $Re\, \sigma > \gamma$

[require $F(\sigma) \to 0$ as $R^{-\varepsilon}$ $\varepsilon \to 0$ so that $F(\sigma)$ ~~is~~ does not grow] and $\int$ along semicircle $\to 0$

cw integral

$$\mathcal{L}[f(t)] = + 2\pi i \cdot \frac{1}{2\pi i} \lim_{\sigma \to s} (\sigma - s) \frac{F(\sigma)}{\sigma - s} = F(s) \, !$$

–ve sign again

Ok so proved that ... but when using can't use loop to right because the e^{st} unchecked by the ~~inverse~~ transform $e^{-\sigma t}$ now would become large on the semicircle. ∴ Must do loop to LEFT! of γ which will now include all those poles!

Example

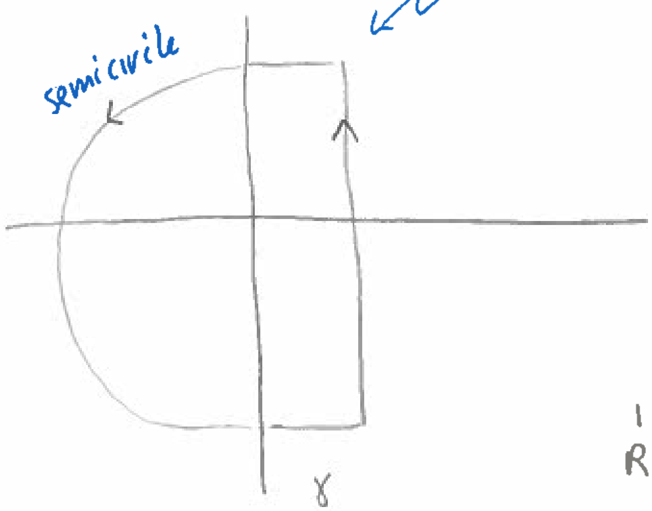
$$F(s) = \frac{3s}{s^2 - 2s - 3}$$

the denominator factors to $(s+1)(s-3)$

so we have simple poles at $s=-1$ and $s=3$

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{3s}{(s+1)(s-3)}\,ds$$

use this contour for integral



on the semicircle ... if $F(s) \to 0$ as $R \to \infty$ (which it does), the $\int$ along semicircle $\to 0$. [Jordan's lemma]

$$\lim_{R\to\infty} \int_{\substack{\text{upper } \frac{1}{2} \text{ circle} \\ |z|=R}} f(z)\, e^{ikz}\, dz \to 0$$

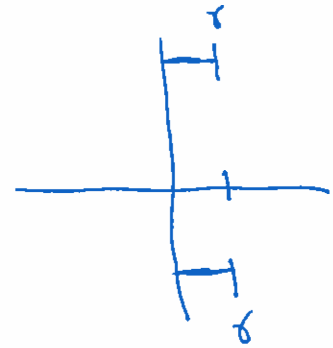
so, don't worry about semi circle

What about segments

$$i\infty < s < \gamma + i\infty$$

&

$$-i\infty < s < \gamma - i\infty$$



they better go → 0

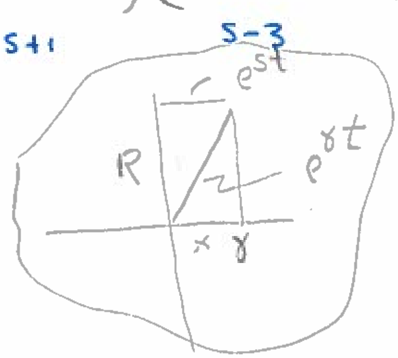
let $\mathrm{Re}(s) = x$ on segment

so we consider $0 \le x \le \gamma$

$$\left|\int_{\text{arc}}\right| \le (\text{length of path}) \cdot (\text{Max of } |\text{integrand}| \text{ on path})$$

$$\int \frac{3se^{st}}{(s+1)(s-3)}\,ds \le \gamma \cdot \underbrace{3e^{\gamma t}} \cdot \max\left|\frac{\overset{s}{x+iR}}{\underbrace{(x+1+iR)}_{s+1}\underbrace{(x-3+iR)}_{s-3}}\right|$$

length of path



$$|e^{st}| \le \underbrace{e^{\gamma t}}_{x \le \gamma} \cdot \underbrace{e^{iRt}}_{|\;|=1}$$

$$\le \gamma\, 3e^{\gamma t} \max\left|\frac{R\sqrt{1+\frac{x^2}{R^2}}}{R^2\left[\left(1+\frac{x^2+2x+1}{R^2}\right)+\left(1+\frac{x^2-6x+9}{R^2}\right)\right]^{1/2}}\right|$$

$$\longrightarrow 0 \text{ as } R \to \infty$$

ok, gone to 0 on semicircle AND on little segments so residue theorem works . . .

$$f(t) = \frac{1}{2\pi i} \cdot 2\pi i \cdot \sum \text{residues}$$

$$f(t) = 2\pi i \cdot \frac{1}{2\pi i} \cdot 3 \left[\frac{-e^{-t}}{-4} + \frac{3e^{3t}}{4} \right]$$

$$f(t) = \frac{3}{4}\left(3e^{3t} + e^{-t}\right)$$

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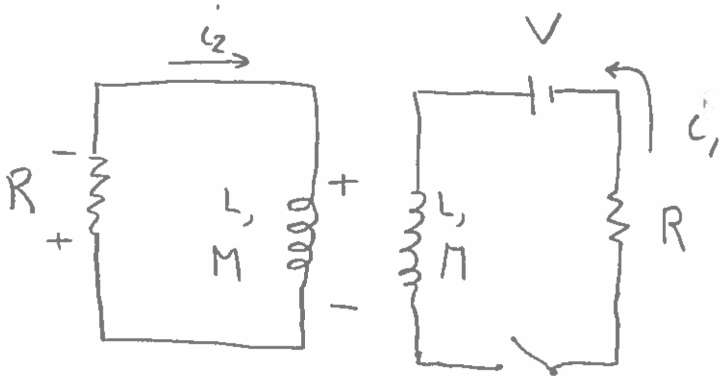
☆ Example of bad mathematical model ☆



p 4820 user
Len Climbs

switch is closed at $t=0$ & $i_1 = i_2 \big|_{t=0} = 0$

For primary circuit

$$V = i_1 R + L\frac{di_1}{dt} + M\frac{di_2}{dt}$$

$$0 = i_2 R + L\frac{di_2}{dt} + M\frac{di_1}{dt}$$

$$\mathcal{L}[\quad] \ldots$$

Laplace space get

$$\frac{V}{s} = I_1 R + sLI_1 - L\cancel{i_1(0)} + sMI_2 - M\cancel{i_2(0)}$$

$$0 = I_2 R + sLI_2 - L\cancel{i_2(0)} + sMI_1 - M\cancel{i_1(0)}$$

approach here ...

$$X \equiv I_1 + I_2$$
$$Y \equiv I_1 - I_2$$

add eqns

$$\frac{V}{s} = XR + sLX + sMX \qquad (1)$$

and subtract

$$\frac{V}{s} = YR + sLY - sMY \qquad (2)$$

because $I_2 - I_1$

Solve for X

$$X = \frac{V}{s} \underbrace{\left[\frac{1}{s(L+M) + R} \right]}$$

suggests integral

$$\mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = \frac{F(s)}{s}$$

primitive fn theorem

$$\frac{1/L+M}{S + R/L+M}$$

a translation

$$\mathcal{L}^{-1} = \frac{1}{L+M} e^{-\alpha t}$$

$$\alpha = R/L+M$$

And the integral

$$X(t) = \frac{V}{L+M} \int_0^t e^{-\alpha t}\, dt = \frac{V}{L+M} \frac{-1}{\alpha} e^{-\alpha t}\Big|_0^t$$

$$= \frac{V}{L+M} \cdot \frac{L+M}{R} \cdot -1\left(e^{-\alpha t} - 1\right)$$

$$\boxed{x(t) = \frac{V}{R}\left(1 - e^{-\alpha t}\right)} \qquad \alpha = \frac{R}{L+M}$$

Similarly for y

$$Y = \frac{V}{S}\left(\frac{1}{R + SL - SM}\right)$$

$$\boxed{y(t) = \frac{V}{R}\left(1 - e^{-\beta t}\right)} \qquad \beta = \frac{R}{L-M}$$

$L \geq M \quad \beta > 0$

(can't have mutual inductance greater than self inductance)

Go back to extract i_1 and i_2

$$i_1 \equiv \frac{X+Y}{2}$$

$$i_1 = \frac{V}{2R}\left[2 - e^{-\alpha t} - e^{-\beta t}\right]$$

$$i_2 = \frac{X-Y}{2}$$

$$i_2 = \frac{V}{2R}\left[e^{-\beta t} - e^{-\alpha t}\right]$$

initial condition? $t=0 \rightarrow$ $i_1 = 0$

$i_2 = 0$

good

Also, $t \rightarrow \infty$ $\quad i_1 \rightarrow \frac{V}{R}$

i_2 is the difference between two small values so ~~afte~~ $\rightarrow 0$

All good

but what if $M \rightarrow L$

(a perfect transformer?)

What if $L = M \Rightarrow \beta \to \infty$
(two perfect windings)

go back to eqn 2

$$\frac{V}{s} = YR + s\cancel{LY} - s\cancel{MY}$$

$$\frac{V}{s} = YR \quad — \quad Y = \frac{V}{Rs} \qquad \boxed{y(t) = \frac{V}{R}}$$

and eqn 1

$$\frac{V}{s} = xR + sLx + sMx$$

$$= xR + 2sLx$$

$$X = \frac{V/s}{2Ls+R} = \frac{V}{s} \; \frac{\frac{1}{2}L}{s + R/2L}$$

integral

translation

$$x(t) = \frac{V}{2L} \int_0^t e^{-\frac{R}{2L}t} \, dt$$

$$= -\frac{V}{\cancel{2L}} \frac{\cancel{2L}}{R} e^{-\frac{R}{2L}t} \Big|_0^\infty$$

$$\boxed{= \frac{V}{R}\left(1 - e^{-\gamma t}\right)} \qquad \boxed{\gamma = \frac{R}{2L}}$$

go back again for i_1 & i_2

$$i_1 = \frac{1}{2}(X+Y) = \frac{V}{2R}\left(2 - e^{-\gamma t}\right)$$

$$i_2 = \frac{1}{2}(X-Y) = \frac{-V}{2R}\, e^{-\gamma t}$$

For big t, $i_1 \to \frac{V}{R}$ and $i_2 \to 0$ ✓

$t \to 0$ $\quad i_1 = \frac{V}{2R}$ and $i_2 = \frac{-V}{2R}$!

Not good !

i_2 cannot change discontinuously (in particular with the inductor in the system).