§ 2.9 $\quad (x+1)! = (x+1)\cdot x!$! $\quad \Gamma 1$

The Γ function has as it most revealing property:

$$\underline{\Gamma(n+1) = n!} \quad !$$

there are a lot of (series definitions (and product) but, the integral definition courtesy of Euler ...

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$$

$$= 2\int_0^\infty \underbrace{e^{-t^2} t^{2z-1}}_{z \text{ complex}}\, dt \, !$$

But $Re\{z\} > 0$!

Easy to evaluate $\Gamma(1)$

$$\Gamma(1) = \int_0^\infty e^{-t} = -e^{-t}\Big|_0^\infty = 1$$

Now integrate the general relation (by parts)

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$$

$$= t^{x-1}(-e^{-t})\Big|_0^\infty - \int_0^\infty (x-1)(-e^{-t}) t^{x-2}\, dt$$

$$= (x-1)\,\Gamma(x-1) \qquad x \geq 1 \;\; >$$

And for integer values ...

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1) = \cdots n(n-1)(n-2)\cdots 1\cdot\Gamma(1)$$

$$= n!$$

Done

Can be extended to all real & complex arg.

The thing about the Γ function is that it is fundamentally defined as a difference !

$$z\,(z-1)! = z!$$

$$z\,\Gamma(z) = \Gamma(z+1) \longleftarrow$$ notice this is in general complex

Euler gives us

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z\,(z+1)\cdot(z+2)\cdots(z+n)}\cdot n^{z}$$

In this definition, replace z with $z+1$

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{1\cdot 2\cdots n}{(z+1)(z+2)\cdots(z+n+1)}\, n^{z+1}$$

$$= \lim_{n\to\infty} \frac{n\,z}{(z+n+1)}\cdot\frac{1\cdot 2\cdot 3\cdots n\; n^{z\cancel{+1}}}{z\,(z+1)(z+2)\cdots(z+n)}$$

$$\Gamma(z+1) = z\,\Gamma(z)$$

notice that for Real arguments this property necesarily popped out of our integral formulation.

Now what of complex values ?

$$\ln\Gamma(z) = z\ln(z) - z - \frac{1}{2}\ln\frac{z}{2\pi} + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260\,z^5}$$

The integral definition is okay for complex arguments so long as $Re(z) > 0$ (need for $\int$ convergence)

~~Also~~ (lower limit diverges $x \leq 0$)

also, $t^{z-1} e^{-t}$ has a branch point at the origin for non-integral z, realising the integral may not bee easy!

Beware!

We then want to consider

$$\oint e^{-t} t^{z-1} \, dt$$

function has trouble at 0 ~~and~~ non integral values ~~on~~ ~~Re(x) axis~~

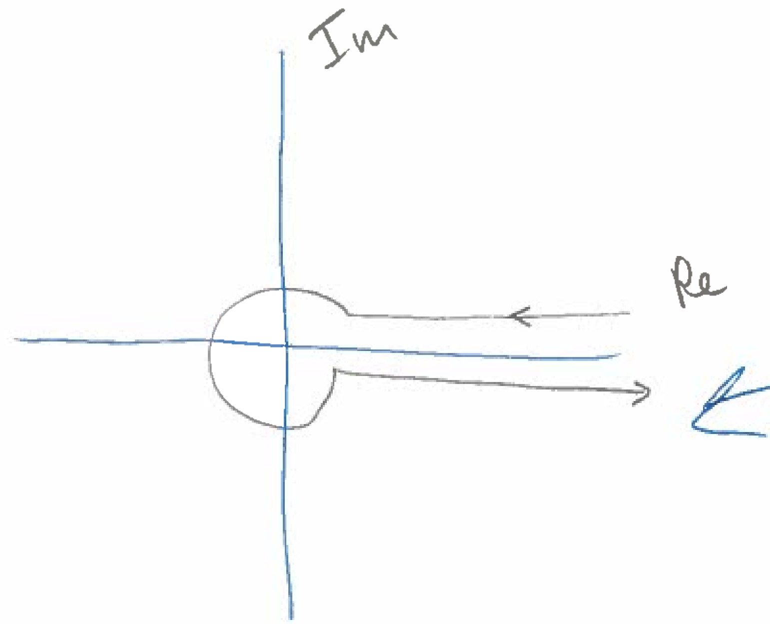
$$f(z,t) = t^{z-1} e^{-t} = e^{\ln(t^{z-1})} e^{-t}$$

$$= e^{(z-1)\ln t - t}$$

given $t = re^{i\theta}$

$$\ln t = \ln(re^{i\theta})$$
$$= \ln r + \ln e^{i\theta}$$
$$= \ln r + i\theta$$

choose integral domain



show domain first!

avoids - real and is consistent with old definition here 4/1/19

On the top of the branch cut, $\theta = 0$, while on the bottom $\theta = 2\pi$.

$$t = re^{i\theta}, \qquad \ln t = \ln r \quad \text{on top}$$
$$= \ln r + 2\pi i \quad \text{on bottom}$$

$$e^{(z-1)\ln t - t} = e^{(z-1)\cdot \ln r - t} \quad \text{on top}$$
$$= e^{(z-1)\cdot(\ln r + 2\pi i) - t} \quad \text{on bottom}$$

$$e^{(z-1)(\ln r + 2\pi i) - t} - e^{(z-1)\ln r - t}$$

these differ by ~~so difference will be~~ $e^{(z-1)\cdot 2\pi i} = \boxed{e^{2\pi i\cdot z}}$ ☆ $\quad e^{2\pi i} = 1!$

This is used again p 6!

Now around the infinitesimal circle: let $|z| = \varepsilon \quad t = \varepsilon e^{i\theta}$

$$\oint t^{z-1}\cdot e^{-t}\,dt = \int_0^{2\pi} \underbrace{(\varepsilon e^{i\theta})^{z-1}}_{t^{z-1}} \cdot \underbrace{e^{-\varepsilon e^{i\theta}}}_{e^{-t}} \cdot \underbrace{\varepsilon i e^{i\theta}\, d\theta}_{dt}$$

$$\lim_{\varepsilon\to 0} \longrightarrow \varepsilon^z \cdot \int_0^{2\pi} e^{i\theta\cdot(z-1)} \cdot 1 \cdot i e^{i\theta}\, d\theta$$

$$\longrightarrow \varepsilon^z \int_0^{2\pi} i e^{iz\theta}\, d\theta = \frac{\varepsilon^z}{z} e^{iz\theta}\Big|_0^{2\pi}$$

$$= \frac{\varepsilon^z}{z}\left(e^{2\pi i z} - 1\right)$$

$$\longrightarrow 0 \quad \text{for } \mathrm{Re}(z) > 0$$

$$e^{(z-1)(\ln r + 2\pi i) - t} - e^{(z-1)\ln r - t}$$

$$e^{(z-1)\ln r - t} \left(e^{2\pi i} - 1\right)$$

$\Gamma 5$

$\theta = 0$ $\theta = 2\pi$

$$\oint = \left[\int_\infty^0 + \int_0^{2\pi} \xi + \int_0^\infty \right] dt \; t^{z-1} e^{-t}$$

$\xi \to 0$ goes to 0

$\int \xi \cdot e^{2\pi i z} - \xi e^0$

$=$ differ by $e^{2\pi i z}, e^0$

flip

$$= \left(e^{2\pi i z} - 1\right) \left[\int_0^\infty e^{-t} t^{z-1} \, dt \right]$$

but actually $z = \mathrm{Re}(z)$ on that segment!

$z \rightleftarrows x$

$$\int_C t^{x-1} e^{-t} \, dt = \left(e^{2\pi i x} - 1\right) \int_0^\infty t^{x-1} e^{-t} \, dt$$

$$= \left(e^{2\pi i x} - 1\right) \Gamma(x)$$

So we can define gamma as

$$\Gamma(z) = \frac{1}{e^{2\pi i z} - 1} \oint_C t^{z-1} e^{-t} \, dt \quad *$$

which decomposes to $\Gamma(x)$ for z Re

notice $\oint$ as opposed $\int_0^\infty$!

$*$ has singularities @ $e^{2\pi i z} - 1 = 0 \Rightarrow z = n$

look at them — singularities that is Γ 6

Consider * with $z = n+\delta$

$$\frac{1}{e^{2\pi i z}-1}\oint t^{n+\delta-1}e^{-t}\,dt = \frac{1}{e^{2\pi i z}-1}\left[\int_\infty^0 t^{n+\delta-1}e^{-t}\,dt + \int_0^\infty e^{2\pi i(n+\delta)}t^{n+\delta-1}e^{-t}\,dt\right]$$

see * p Γ 4

$$= \frac{e^{2\pi i(n+\delta)}-1}{e^{2\pi i(n+\delta)}-1}\int_0^\infty t^{n+\delta-1}e^{-t}\,dt$$

would become $\frac{0}{0}$ in limit!

$$= \int_0^\infty t^{n+\delta-1}e^{-t}\,dt \qquad \lim_{\delta\to 0}$$

$$\longrightarrow \Gamma(n) \checkmark$$

also has singularities at $-$ve integers ...
can't get rid of them.

Why solve analytically at all?

Laplace Transform

A "linear" integral transform that can convert a differential problem into an algebraic problem!

It works well with terms of the form $e^{\alpha t}$ and by extension then to sin, cos etc., Impulsive forcing terms, and it explicitly hands you solution for given initial conditions.

$$\begin{array}{ccc} \text{Linear DE} & \xrightarrow[\text{Transform}]{\text{Laplace}} & \text{Algebraic equation} \\ y(t) & & \text{in } Y(s) \\ \downarrow \text{ugly} & & \downarrow \text{algebraic soln. (pretty)} \\ y(t) = \cdots & \xleftarrow[\text{inverse transform}]{\text{(black magic)}} & Y(s) = \\ & & * \end{array}$$

$$\boxed{\mathcal{L}(f) = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt}$$

Laplace Transform

works ~~best~~ for non homogenous term with a discontinuous nature

transients in E circuits!

7/1/19 →

? does the integral converge? $\Rightarrow$ $e^{-st} \to 0$ as $t \to \infty$ pretty aggreively!

|| Laplace transform exists $\forall$ piecewise continuous functions of "exponential order".

exponential order:

A function f is of exponential order σ_0 if there exist a real positive constant M such that

$$\left| e^{-\sigma_0 t} \cdot f(t) \right| \leq M \qquad \forall t \quad 0 < t < \infty$$

σ_0 real.

f is then said to be of exponential order σ_0

$$\left| e^{-\sigma_0 t} \quad e^{3t} \right| = e^{(3-\sigma_0)t}$$

$$\lim_{t \to \infty} e^{(3-\sigma_0)t} \to 0 \qquad \text{for } \sigma_0 > 3$$

$$\to = 1 \qquad \text{for } \sigma_0 = 3 \Leftarrow$$

$$\to \infty \qquad \text{for } 0 < \sigma_0 < 3 \qquad 0 < \sigma_0 < 3$$

makes sense e^{3t} behaves as ... e^{3t} problem?

So, what's the transform of an exponential?

$$f(t) = e^{\alpha t}$$

$$F(s) = \int_0^\infty e^{\alpha t} e^{-st} \, dt = \frac{1}{\alpha - s} e^{(\alpha - s)t} \Big|_0^\infty$$

for $Re(s) > \alpha$!

$$= \frac{1}{\alpha - s} \left[0 - 1 \right]$$

$$\boxed{F(s) = \frac{1}{s - \alpha}} \to (s > \alpha)$$

Powers $f(t) = t^m$

is it exponential order?

recall taylor series for exponential

$$e^{\varepsilon t} = 1 + \varepsilon t + \frac{\varepsilon^2 t^2}{2} + \cdots \frac{\varepsilon^m t^m}{m!}$$

$$\frac{\varepsilon^m t^m}{m!} < e^{\varepsilon t}$$

$$\Rightarrow \quad t^m < \frac{m!}{\varepsilon^m} e^{\varepsilon t}$$

$$\left| t^m e^{-\varepsilon t} \right| < \frac{m!}{\varepsilon^m} e^{-\varepsilon t} e^{\varepsilon t} = \frac{m!}{\varepsilon^m}$$

just a number! M

for true real ε these functions are of exponential order ε

for $m=0$ the transform: ie $t^0 = 1$

$$\mathcal{L}(1) = \int_0^\infty e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s} \qquad Re(s) > 0$$

for $m \geq 1$ ... start with $m=1$

$m=1$ $$\mathcal{L}(t) = \int_0^\infty t e^{-st}\,dt \quad \Rightarrow \quad \int u\,dv = uv - \int v\,du + c$$

$dv = e^{-st}$ $\quad u = t$

$$\int dv = -\frac{e^{-st}}{s}$$

integrate by parts → goes $0-0$

$$= \cancel{\frac{-t e^{-st}}{s}}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= \frac{-e^{-st}}{s^2}\Big|_0^\infty = \frac{1}{s^2} \qquad (Re(s) > 0)$$

okay now for $m > 1$

$$\mathcal{L}(t^m) = \int_0^\infty t^m e^{-st}\,dt = \cancel{\frac{t^m e^{-st}}{s}}\Bigg|_0^\infty +$$

$$+ \frac{1}{s}\int_0^\infty m t^{m-1} e^{-st}\,dt$$

$$= \frac{m}{s}\mathcal{L}(t^{m-1}) = \frac{m(m-1)}{s^2}\mathcal{L}(t^{m-2})$$

12/1/07

$$\boxed{\mathcal{L}(t^m) = \frac{m!}{s^{m+1}}}$$

another approach on this now we see the factorial come into play we recall the gamma function:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt$$

with the astounding property that $\Gamma(n+1) = n!$!

let $u = st$

$$\mathcal{L}(t^p) = \int_0^\infty t^p e^{-st}\,dt = \int_0^\infty \left(\frac{u}{s}\right)^p e^{-u}\,\frac{1}{s}\,du$$

$$= \frac{1}{s^{p+1}}\int_0^\infty u^p e^{-u}\,du = \frac{1}{s^{p+1}}\Gamma(p+1)$$

$$\boxed{\mathcal{L}(t^p) = \frac{1}{s^{p+1}}\Gamma(p+1)}$$

$\mathrm{Re}(s) > 0$
$p > 0$

and now, p is not constrained to be an integer!

sines and cosines will follow because they are linear combinations of exponentials!

$$\mathcal{L}(\cos \omega t) = \int_0^\infty \left[\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right] e^{-st}\, dt$$

$$= \frac{1}{2}\left[\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right] = \frac{s}{s^2+\omega^2} \qquad Re(s) > 0$$

Collect Results

$f(t)$	$F(s)$	constraint
e^{at}	$\frac{1}{s-a}$	$s > \alpha$
1	$\frac{1}{s}$	$s > 0$
t^p	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$s > 0$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$

Taking transforms is easy!

Inverting them is not, you need to rely on the properties of the transform and a knowledge of the standard transforms!

In general the transform is not unique ... not generally a problem for "physical" situations because you can expect continuous functions.

Properties of the Laplace transform

1) $\mathcal{L}$ is linear

$$\mathcal{L}(af) = \int_0^\infty a \, f(t) e^{-st} \, dt = a \int_0^\infty f(t) e^{-st} \, dt$$

$$= \boxed{a \, \mathcal{L}(f) \; !}$$

$$\mathcal{L}(f_1 + f_2) = \int_0^\infty (f_1 + f_2) e^{-st} dt = \boxed{\mathcal{L}(f_1) + \mathcal{L}(f_2) \; !}$$

2) transform of derivative

$$\mathcal{L}\left(\frac{df}{dt}\right) = \int_0^\infty \underbrace{\frac{df}{dt}}_{dV} \underbrace{e^{-st}}_{U} \, dt$$

$$= \underbrace{f}_{V} \underbrace{e^{-st}}_{U} \Big|_0^\infty - \int_0^\infty (-s) f(t) e^{-st} dt$$

notice explicit b.c.

$$\boxed{\mathcal{L}\left(\frac{df}{dt}\right) = -f(0) + s \, F(s)}$$

only valid for $Re(s) > s_0$! iff f is of exponential order s_0

the transform of higher derivatives can be found by repeating the process.

$$\boxed{\mathcal{L} \frac{d^m f}{dt^m} = s^m F(s) - \sum_{n=1}^{m} \frac{d^{m-n} f}{dt^{m-n}}\Big|_0 s^{n-1}}$$

3) attenuation property

↳ $e^{-\alpha t}$

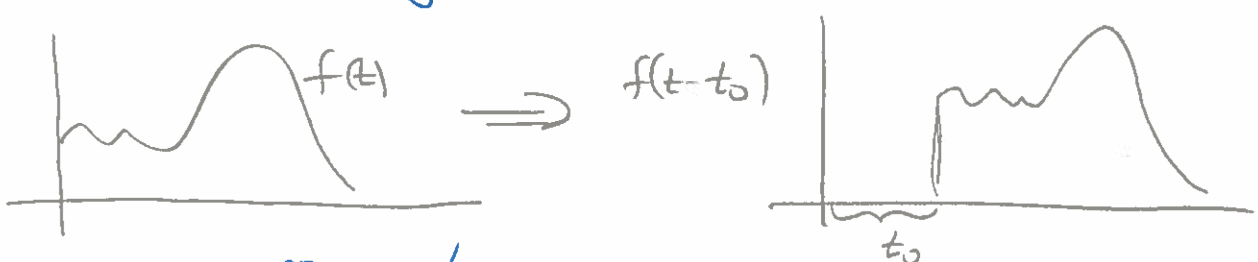
$$\mathcal{L}\left(e^{-\alpha t} f(t)\right) = \int_0^\infty e^{-\alpha t} f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty f(t)\, e^{-(s+\alpha)t}\, dt$$

$$\boxed{\mathcal{L}\left(e^{-\alpha t} f\right) = F(s+\alpha)}$$

↳ attenuation transforms into a translation in s space!

4) shifting property (hmm recall the attenuation)



problem, $\int_0^\infty = \int_0^{t_0} + \int_{t_0}^\infty$

↳ what to do?

hammer with step function

$$f(t-t_0) \equiv S(t-t_0)\, f(t-t_0)$$

ignoring bit when $\propto t < t_0$ because $S = 0$!

$$\mathcal{L}\left[S(t-t_0)\, f(t-t_0)\right] = \int_0^\infty S(t-t_0)\, f(t-t_0)\, e^{-st}\, dt = \int_{t=t_0}^\infty f(t-t_0) e^{-st} dt$$

let $u = t - t_0$ and $\int$ over u

$$= \int_0^\infty f(u)\, e^{-(u+t_0)s}\, du$$

$t = t_0 \Rightarrow u = 0$

$$= e^{-st_0} \int_0^\infty f(u)\, e^{-us}\, du = e^{-st_0} F(s)$$

$$\boxed{\mathcal{L}\left[S(t-t_0)\, f(t-t_0)\right] = e^{-st_0} F(s)} \qquad t_0 > 0$$

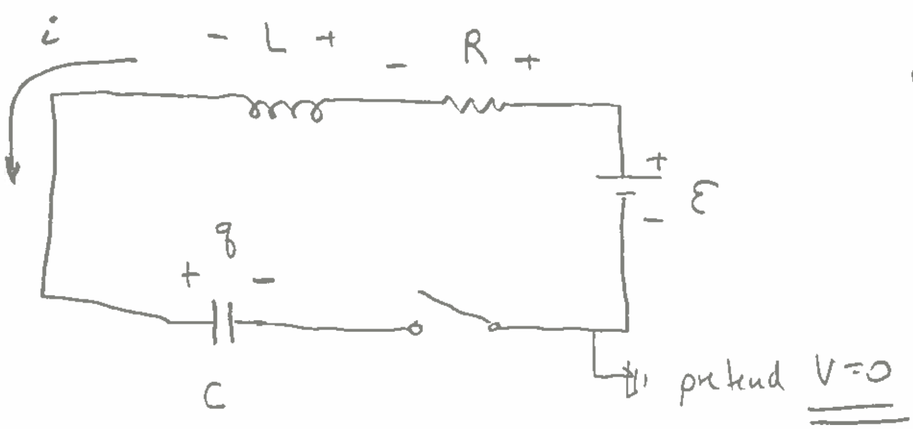
so ~~similarly~~ related to the attenuation a translation becomes an attenuation in s space.

end 12/1/07

Now What

start 15/1/07

Lets solve a problem, the only problems I am aware of that regularly use Laplace transforms are transient analysis in electronic circuits ...



$i = \frac{dq}{dt}$

$V = iR$

$V = L\frac{di}{dt}$

$CV = q$

What is the current for $t>0$ if the switch is closed at $t=0$?

Notice → this question plays into Laplace analysis because it has a well defined time start!

- These questions ~~always~~ often require that the switch has been open for a long time (initially) so that $i\big|_{t=0}=0$ and we will take the capacitor as uncharged $q=0$.

Use Kirchoffs rule $\qquad i = \frac{dq}{dt}$

$$\frac{q}{C} + L\frac{di}{dt} + iR - \varepsilon = 0$$

(signs are very critical, think about what you are doing)

$\mathcal{L}\left[\frac{d^2 f}{dt^2}\right] = s^2 F(s) - s\,f(0) - \left.\frac{df}{dt}\right|_0$

9a

$$\frac{q}{C} + L\frac{di}{dt} + iR - \mathcal{E} = 0$$

$$\frac{q}{C} + L\frac{d^2q}{dt^2} + \frac{dq}{dt}R - \mathcal{E} = 0$$

$\mathcal{L}[q] = Q$
$\mathcal{L}[i] = I$

$$\mathcal{L}\left[\qquad\qquad\right] = 0$$

$$\frac{1}{C}Q + L\left[-\underbrace{\left.\frac{dq}{dt}\right|_0}_{=0} - s\underbrace{q_0}_{=0} + s^2 Q\right] + \left[-q_0 + sQ\right]R + -\frac{\mathcal{E}}{s} = 0$$

a constant

$$\frac{Q}{C} + Ls^2Q + sQR = \frac{\mathcal{E}}{s}$$

$\mathcal{E}$ $\frac{di}{dt}$ would have to be ∞ to create the needed discontinuity in q

q

$t=0$

$$Q\left[\frac{1}{C} + Ls^2 + sR\right] = \frac{\mathcal{E}}{s}$$

$$Q = \frac{\mathcal{E}}{s}\cdot\frac{1}{\frac{1}{C} + Ls^2 + sR}$$

$$= \frac{\mathcal{E}}{sL}\cdot\frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

? how to invert ?

look for sin/cos $\frac{a}{a^2 + s^2}$

look at denominator

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = \left(s + \frac{R}{2L}\right)^2 + \frac{1}{LC} - \frac{1}{4}\frac{R^2}{L^2}$$

completing square

$$s^2 + \frac{R^2}{4L^2} + \frac{sR}{L} + \frac{1}{LC} - \frac{1}{4}\frac{R^2}{L^2} \quad \checkmark$$