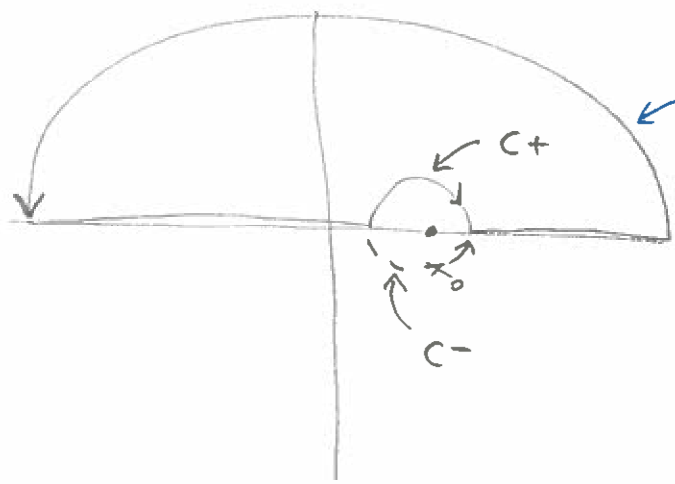
Occasionally an isolated pole will be directly on the contour of integration causing the integral to diverge → Let us illustrate with a physical case →

Arfken p458

Consider $f(x)$ with pole on x-axis at x_0



for now go ccw

Now p 514 new 7th edition BUT changed development — matches Lea's!

I want to integrate $\int_{-\infty}^{\infty} f(x)\,dx$!

we introduce a small semi-circular detour around $x = x_0$ in that area

$$z - x_0 = \delta e^{i\varphi}$$

$$dz = i\delta e^{i\varphi}\,d\varphi$$

so the integral contribution is:

ccw below $\quad \int_{C^-} a_{-1}\frac{dz}{z-x_0} = a_{-1} i \int_{\pi}^{2\pi} \frac{\cancel{\delta e^{i\varphi}}\,d\varphi}{\cancel{\delta e^{i\varphi}}} = i\pi a_{-1}$

contribution

residue (the a_{-1} coefficient of the Laurent series)

cw above $\quad \int_{C^+} a_{-1}\frac{dz}{z-x_0} = a_{-1} i \int_{\pi}^{0} \frac{\cancel{\ }}{\cancel{\ }}\,d\varphi = -i\pi a_{-1}$

$\frac{1}{2}$ the residue theorem

Now consider the residue theorem:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{C} f(z)\,dz$$

$$= \int_{-\infty}^{x_0-\delta} f(x)\,dx + \int_{C_{x_0}} f(z)\,dz + \int_{x_0+\delta}^{\infty} f(x)\,dx + \int_{\substack{\infty \\ \text{semi} \\ \text{circle}}} f(z)\,dz$$

If C_{x_0} includes x_0 (we go below the x-axis)
ccw
x_0 is enclosed

$$\oint f(z)\,dz = 2\pi i a_{-1} = \int_{-\infty}^{x_0-\delta} + i\pi a_{-1} + \int_{x_0+\delta}^{\infty} + \int_{\substack{\infty \\ \text{sem}}} f(z)\,dz$$

If C_{x_0} excluded x_0 (we go above

$$\oint f(z)\,dz = 0 = \int_{-\infty}^{x_0-\delta} - i\pi a_{-1} + \int_{x_0+\delta}^{\infty} + \int_{\infty} f(z)\,dz$$

both ways for this case

$$\boxed{\oint f(z)\,dz = \pi i a_{-1}}$$

Cauchy Principal Value

pole on path counts half ✱